Mathematical Reviews

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LOGIC AND FOUNDATIONS

See also 263.

1:

✱**Meschkowski, Herbert. Wandlungen des mathematischen Denkens. Eine Einführung in die Grundlagenprobleme der Mathematik.** Friedr. Vieweg & Sohn, Braunschweig, 1956. iii+122 pp. DM 12.80.

The book has as a general aim to stimulate the mathematical student of the first year, and other mathematicians, to engage in philosophical problems posed by mathematical foundations research. It also attempts to awaken understanding for the "mathematical way of thought" in students of other branches. It must, however, be said that the book is probably understandable only for mathematically gifted students.

The book comprehends the following chapters: The problem; The foundations of Greek mathematics; The way to non-Euclidean geometry; The problematics of the infinite; Cantor's establishment of the theory of sets; Antinomies and paradoxes; Intuitionism; Geometry and experience; Problems of mathematical logic; Formalism; Problems of decision; The philosophical output of mathematical foundations research.

The book is written in a scientific rather than philosophical style. It is a merit of the author that he includes abundant quotations from the mathematicians and philosophers in question.

We mention here some items from the last chapter of the book. According to the author, the characteristic feature connecting all the modern researches about the foundations of mathematics is that they endeavor to avoid metaphysical elements in the building up of mathematics. There is no place for the "absolute" in mathematics; it is only an "enthusiastical exaggeration". There results thus a rejection of Platonism. Even Hilbert, who has put on mathematics absolutistic claims, wishes not the "truth" from mathematics, but only "security", namely the security that there cannot be deduced in a formal system together with the formula f also its negation $\bar{f}$.

But are there only positivists among modern mathematicians? asks the author. His answer is no. The author takes an intermediate position. On the one side there are compelling reasons for turning away from metaphysical elements in the building up of mathematics: an ontology of the basic geometrical conceptions has proved impossible; the theory of sets has led to antinomies; and the unlimited use of the "tertium non datur" is not reconcilable with mathematical arguments. On the other hand, the entire expulsion of metaphysical elements from mathematics is not possible. Curry has said of the intuitionists that they believe in the God "intuition". And Steigmuller has shown that the logistical attempt at the establishment of mathematics does not renounce "abstract Platonical entities". He says that a self-guarantee of human thought is excluded. One cannot gain a positive result without any presupposition. One must already believe in something, in order to justify something else. But these formulations seem not entirely right to the author, because they efface the difference between the "evident" in mathematics and metaphysics. The evident in mathematical logic is the basis for a consistent argumentation. But against the "evidence-presuppositions" in metaphysical argumentations there is always possible the objection that there is question only about "superfluous habits of thinking".

The author discusses also the views of K. Reidemeister. It appears, according to the author, as a particular task of the mathematician and the (natural) scientist to point out the significance of "Sachlichkeit" in a time in which the whole world puts the "existence before the essence" (according to Sartre's formula).

As we see, the author is anxious for a maximum of objectivity in his explanations. *B. Germansky* (Berlin)

2:

✱**Ladrière, Jean. Les limitations internes des formalismes. Etude sur la signification du théorème de Gödel et des théorèmes apparentés dans la théorie des fondements des mathématiques.** Collection de Logique Mathématique, Série B, II. E. Nauwelaerts, Louvain; Gauthier-Villars, Paris, 1957. xiv+715 pp. 650 francs belges.

This book contains in 400 pages a treatment of the negative results in metamathematics with detailed proofs of the main theorems. It covers the subject almost completely. In most cases the author gives the proofs in their original form, but he surrounds them by many clarifying remarks. Furthermore, there are about 50 pages of philosophical speculation, 50 of complements to the main text, 50 of bibliography and 150 of tables. The first two chapters, which are introductory, contain the necessary notions on formal systems, metamathematics, the logical paradoxes and recursive functions. Here follows a survey of the main subjects which are treated in the other chapters. Ch. 3–5: Gödel's incompleteness theorem and related results. Rosser's extensions to simply consistent systems and to non-recursive logics. Impossibility of consistency proofs; consequences for Hilbert's Beweistheory. Sketch of Gentzen's consistency proof for arithmetic. Ch. 6: Church's theorem on the unsolvability of the decision-problem. Kleene's theorem that the class of recursive functions is not recursively enumerable. Turing's theorem that a general decision-machine in his sense is impossible. Post's theorem on the unsolvability of the decision-problem for his normal systems. Summary of results on the word problem and on degrees of insol-

vability. Ch. 7: Kleene's hierarchy of predicates; his proofs of Gödel's and Church's theorems. Extension to non-recursive logics. Ch. 8: Fundamental notions of semantics. Semantical incompleteness theorems of Tarski and Mostowski. Ch. 9: The Löwenheim-Skolem theorem. Henkin's theorems on non-standard models. Combinatory logic; combinatory and deductive saturation; theorem of Kleene-Rosser-Curry. Systems of Myhill and Church for which Gödel's and Tarski's theorems are not valid.

A. Heyting (Amsterdam)

3:

Robinson, Raphael M. Restricted set-theoretical definitions in arithmetic. Proc. Amer. Math. Soc. **9** (1958), 238–242.

A restricted set-theoretical definition is one in which quantification is allowed only over variables for natural numbers and variables for sets of natural numbers. The usual logical notions, including identity, are permitted, as well as the membership-relation between number-variables and set-variables. Tarski has asked: (I) Is there a restricted set-theoretic definition of addition in terms of successor? (II) Is there a decision method for the theory based on the notion of successor and using restricted set theory? A positive answer to (I) would imply a negative answer to (II). The author gives a positive answer to a weaker form of (I) by providing a restricted set-theoretical definition of addition in terms of the operations of successor and double, and hence also a negative answer to the corresponding weaker form of (II). He also gives a positive answer to (I) for a certain special model of the theory of the successor operation.

E. Mendelson (New York, N.Y.)

4:

Kalmár, László. A direct proof of unsolvability by a general algorithm of decision problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 1–25. (Hungarian)

An exposition of the German article in Z. Math Logik Grundlagen Math. **2** (1956), 1–14 [MR **18**, 369].

5:

Mycielski, Jan. A characterisation of arithmetical classes. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1025–1027, LXXXVI. (Russian summary)

The sentences of a given applied first order calculus which are in prenex normal form can be classified according to number and length of alternate blocks of universal and existential quantifiers in the prefix. Let σ be a particular type of sentences defined in this way, and let K and L be two classes of relational systems (structures). The author states a necessary and sufficient semantic condition for all sentences of type σ which hold in K to hold also in L. The proof and some applications are to be published in Fund. Math.

A. Robinson (Jerusalem)

6:

Harrop, R. On the existence of finite models and decision procedures for propositional calculi. Proc. Cambridge Philos. Soc. **54** (1958), 1–13.

A model of a propositional calculus is a truth table interpretation under which the axioms are valid and the rules of inference are preserved (in the sense that, if all premises in an application of a rule are valid, then so is the conclusion). For a "strong" model, if all the premises in an application of a rule have designated values, then so does the conclusion. A propositional calculus has the "finite model property" if every unprovable formula is not valid in some finite model.

The following fundamental results are presented: (1) Given a finite model of a propositional calculus, one can effectively construct a finite strong model that has the same set of valid formulas. (2) If a propositional calculus has the finite model property, then it is decidable. (3) A propositional calculus can be decidable without having the finite model property.

The existence of an undecidable propositional calculus makes (2) a non-trivial result. An undecidable propositional calculus was described, without detailed proof, by Linial and Post in an abstract [Bull. Amer. Math. Soc. **55** (1949), 50]; the author gives an original, detailed proof in the present paper.

(3) is especially significant when compared with the situation for infinite models: for every propositional calculus, there is an infinite model in which all unprovable formulas are not valid. In proof of (3), the author exhibits two decidable propositional calculi without the finite model property. Each example has three axioms and three rules. In the first example, one of the rules is redundant; in the second, axioms and rules are independent.

G. F. Rose (Santa Monica, Calif.)

7:

Mostowski, A. On recursive models of formalised arithmetic. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 705–710, LXII. (Russian summary)

The author considers models of first-order systems of arithmetic, the individuals of the model being the natural numbers or, equivalently, any recursive set of objects. His main result (*) is this: suppose the system contains a sufficient number of primitive symbols so that (i) two disjoint, recursively enumerable, but not recursively separable sets can be defined in the form $(Ex)A(x, n)$, $(Ex)B(x, n)$ with A and B quantifier-free, and (ii) the relevant properties of these sets (disjointness, calculability of $A(n, m)$, $B(n, m)$ for numerals n and m) can be formally proved in the system. Then every recursive model, i.e., one in which the non-logical constants are replaced by recursive functions and predicates, is isomorphic to the standard model. An analogous result is established for recursively enumerable models, and, by a remark of Putnam [J. Symb. Logic **22** (1957), 110–111] also for one number quantifier models with recursive scope. Essentially the same results are contained in the reviewer's abstract [ibid. **22** (1957), 109]. From (*) follows neatly a negative solution of a problem by Vaught: let T_V be the class of first order formulae which are satisfiable (over the natural numbers) by predicates from a class V of 'constructive' predicates, i.e., all predicates of V are recursively enumerable and every primitive recursive predicate is in V; is T_V arithmetically definable for some such V?

The proof of (*) shows that there is no recursive non-standard model in which all theorems of (quantifier-free) primitive recursive arithmetic are valid. In other words, recursive models are useless for independence problems in full primitive recursive arithmetic. This answers a question raised by Skolem at Amsterdam in 1954 [cf. also 'Mathematical interpretation of formal systems', North-Holland Publ. Co., Amsterdam, 1955, pp. 1–14; MR **17**, 699]. It is not known if elementary arithmetic with addition and multiplication as the sole non-logical constants has a recursive non-standard model.

G. Kreisel (Reading)

SET THEORY

See also 272.

8:

Ciorănescu, Nicolas. Le procédé de la dichotomie et le système dyadique dans la théorie des ensembles linéaires bornés. Bul. Inst. Politech. Iaşi. 4 (1949), 337–339.

9:

Kurepa, G. On a new reciprocity, distribution and duality law. Pacific J. Math. 7 (1957), 1125–1143.

Let $\perp$ denote either $\cap$ or $\cup$, and $\perp'$ the other symbol. Let $(e, \perp, f) = \perp'_{e_1} \perp_{e_0} f(e_0)$, where $e_0 \in e_1 \in e$, and f is a set function. For a set function f, by f' is meant the function $f(e_0)'$ for each e_0. Given two families of sets e and e^*, write $(e, e^*) \in (j)$ if $e_1 \cap e_1^* \neq \emptyset$ for all e_1 in e and e_1^* in e^*; and $(e, e^*) \in (k)$ if, whenever $X \cap e_1 \neq \emptyset$, e_1 in e, it follows that $2^X \cap e^* \neq \emptyset$. Fundamental Theorem: The following four statements are equivalent. I. $(e, e^*) \in (j) \wedge (k)$; II. For each mapping f, $(e, \cap, f)' = (e^*, \cap, f')$; III. For each mapping f, $(e, \cap, f) = (e^*, \cup, f)$; IV. $(e^*, e) \in (j) \wedge (k)$.

The author next applies his concepts and results to maximal chains and maximal antichains. Let S be any partially ordered set. OS designates the system of all maximal chains in S. $\bar{O}S$ designates the system of all maximal antichains in S. $O'S$ designates the system of all X in $\bar{O}S$ such that $X \cap M \neq \emptyset$ for M in OS. $\bar{O}'S$ designates the system of all X in OS such that $X \cap A \neq \emptyset$ for A in $\bar{O}S$. It is then shown that $(OS, O'S)$ and $(\bar{O}S, \bar{O}'S)$ are in (j). To apply the Fundamental Theorem to $(OS, O'S)$ and/or $(\bar{O}S, \bar{O}'S)$ it is thus necessary to know that $(OS, O'S)$ and/or $(\bar{O}S, \bar{O}'S)$ are in (k). This is shown for various classes of S. Say that (k) is satisfied for S if $(OS, O'S)$ is in (k). The author then introduces the symbol $(\bar{k})$ but does not give a clear-cut definition of it. From the context it appears that "$(\bar{k})$ is satisfied for S" means that $(\bar{O}S, \bar{O}'S)$ is in (k). Theorem: (a) Condition (k) is satisfied for S when S satisfies one of the following: (1) S is a chain; (2) the elements of OS are pairwise disjoint; (3) S is a ramified table. (b) Condition $(\bar{k})$ is satisfied for S when S satisfies one of the following: (1) S is an antichain; (2) the elements of $\bar{O}S$ are pairwise disjoint; (3) S is a ramified table.

Finally, a general duality law is proved. For each partially ordered set S in (k), $(\Omega, \perp, f)' = (\Omega', \perp, f')$, where Ω is either OS or $O'S$ and Ω' is the other. If $(\bar{k})$ is satisfied for S, then $(\Omega, \perp, f)' = (\Omega', \perp, f')$, where Ω is either $\bar{O}S$ or $\bar{O}'S$ and Ω' is the other.

{It is the opinion of the reviewer that this paper was not refereed with the care and attention usually associated with a first class journal. For example, on page 1132, line 12, there occurs the statement, "It is probable that $O'\sigma_0 = v$." The word "probable" also occurs on page 1139, line 20. Because of this, one gets the impression that in both instances the word "provable" is meant. There are several other instances where the meaning of statements is not clear.} *S. Ginsburg* (Hawthorne, Calif.)

COMBINATORIAL ANALYSIS

See 253, 254, 255.

ORDER

See also 41, 452.

10:

Kimura, Naoki. Total orderings on a semilattice. Proc. Japan Acad. 34 (1958), 124–125.

Necessary and sufficient conditions are given for it to be possible to define a simple ordering in a semi-lattice, such that $a < b$ implies $ac \leq bc$.

G. Birkhoff (Cambridge, Mass.)

11:

Fryer, K. D.; and Halperin, Israel. On the coordinatization theorem of J. von Neumann. Canad. J. Math. 7 (1955), 432–444.

Als Verallgemeinerung des Satzes, daß sich jede n-dimensionale projektive Geometrie für $n \geq 3$ durch homogene Koordinaten aus einem Schiefkörper algebraisieren lässt, bewies J. v. Neumann [Proc. Nat. Acad. Sci. U.S.A. 22 (1936), 92–100, 101–108, 707–713; 23 (1937), 16–22, 341–347] folgenden Satz: Sei L ein komplementärer modularer Verband mit einer homogenen Basis $a_1, \cdots, a_n$, $n \geq 4$; sei L_{ij} ($i \neq j$, i, j beliebig fest) die Menge der Inversen von a_i bezüglich $a_i + a_j$. Dann gilt: (1) Zwischen den Elementen von L_{ij} lässt sich eine Addition und eine Multiplikation so definieren, dass L_{ij} ein regulärer Ring R mit Einselement wird und daß der Teilverband $L(a_i)$, der aus allen $x \leq a_i$ besteht, isomorph ist zum Verband der Linkshauptideale von R; (2) L ist isomorph zu dem Verband aller Linkshauptideale des Ringes aller (n, n) Matrizen mit Elementen aus R. — Zu (2) ist die Aussage äquivalent (2'): L ist isomorph zum Verband aller von endlich vielen Vektoren aufgespannten Linksmoduln im Raum V der Vektoren $(\alpha^1, \cdots, \alpha^n)$, mit α^i aus R. Verff. geben einen direkten Beweis von (2') ohne Benutzung von Matrizenringen oder Erweiterungstheoremen.

R. Moufang (Frankfurt a.M.)

12:

Fryer, K. D.; and Halperin, I. The von Neumann coordinatization theorem for complemented modular lattices. Acta Sci. Math. Szeged 17 (1956), 203–249.

Für den in vorstehendem Referat zitierten Satz von v. Neumann in der Formulierung der Aussagen (1) und (2') geben Verff. eine vereinfachte Darstellung, die für jeden komplementären modularen Verband L mit homogener Basis der Ordnung $n \geq 3$ gilt; für den Fall $n = 3$ werden an den normierten Rahmen von L drei Zusatzbedingungen gestellt, die im Falle $n \geq 4$ von selbst erfüllt sind. Sie bedeuten in einer projektiven Ebene einen Schnittpunktsatz über axiale Vierecke, der mit dem Satz von Desargues äquivalent ist. Damit ist die vollständige verbandstheoretische Verallgemeinerung zu dem Satz über die Algebraisierbarkeit einer projektiven Geometrie der Dimension ≥ 2 geleistet. — Die Arbeit gibt eine ausführliche Behandlung aller benötigten Begriffe und Sätze und ist unabhängig von der einschlägigen Literatur verständlich.

R. Moufang (Frankfurt a.M.)

13:

Balachandran, V. K. On certain BS-representations and a characterization of complete Boolean algebras. Proc. Indian Acad. Sci. Sect. A. 45 (1957), 35–46.

Given a family $\mathfrak{F}$ of dual ideals I_α in a lattice L, a "BS-representation" is a mapping $\mathfrak{F}: a \to \mathfrak{F}(a)$ of each $a \in L$ onto the set of those $I_\alpha \in \mathfrak{F}$ which contain a. It is shown that the following conditions on L are equivalent: (i) one such "BS-representation" is isomorphic; (ii) there exists a family of completely prime dual ideals which

separates L; (iii) every $a \in L$ is expressible as a meet of meet-prime elements of the completion of L by cuts. It is stated without proof that a Boolean algebra is atomic if and only if it has an isomorphic BS-representation which preserves arbitrary joins.

G. Birkhoff (Cambridge, Mass.)

GENERAL ALGEBRAIC SYSTEMS

See also 74.

14:

Ádám, András. A theorem on algebraic operators in the most general sense. Acta Sci. Math. Szeged **18** (1957), 205–206.

Let sets H and $\mathfrak{H}$ be given and an operation $a\mathfrak{a}$ ($a \in H$, $\mathfrak{a} \in \mathfrak{H}$) be defined with values in H. If $\mathfrak{A}$ is a subset of $\mathfrak{H}$ and G a subset of H, then G is $\mathfrak{A}$-replete if G is exactly the set of all elements of H left invariant by the set of those elements of $\mathfrak{A}$ which leave fixed the elements of G. A subset $\mathfrak{G}$ of $\mathfrak{H}$ is $\mathfrak{A}$-replete under dual circumstances. The paper presents conditions under which the property of being $\mathfrak{A}$-replete implies the property of being $\mathfrak{B}$-replete, where $\mathfrak{B}$ is a subset of $\mathfrak{A}$.

J. E. Whitesitt (Bozeman, Mont.)

THEORY OF NUMBERS

See also 58, 428, 449.

15:

Succi, Francesco. Una generalizzazione delle funzioni aritmetiche completamente moltiplicative. Rend. Mat. e Appl. (5) **16** (1957), 255–280.

Let G be the group of arithmetical functions f with $f(1) \neq 0$ and with the convolution product as group operation. A formula of Cipolla [Revista de Math., Torino, **9** (1908)] says that for strongly multiplicative f the inverse in G is equal to the ordinary product $\mu \cdot f$, where μ is the Möbius function. The present author discusses the the class C' of all functions $f \in G$ having this property. He shows inter alia that those values $f(n)$ for which n is the product of an odd number of different primes may be prescribed arbitrarily, and that if $f \in C'$, $f(1)f$ is strongly multiplicative whenever it is multiplicative for these values of n. The paper uses the terminology and notations introduced by Pellegrino [Atti del 4º Congresso dell' Unione Matematica Italiana, v. 2, Taormina, 1951, pp. 161–168; Rend. Mat. e Appl. (5) **15** (1956), 469–504; MR **15**, 779; **19**, 16].

C. G. Lekkerkerker (Amsterdam)

16:

Comment, Pierre. Sur la structure du groupe des unités de l'anneau des fonctions arithmétiques. Rend. Mat. e Appl. (5) **16** (1957), 250–254.

The author considers the set of arithmetical functions $f(n)$ satisfying (i) $f(1) \neq 0$, (ii) $f(nm) = f(n)f(m)/f(1)$ for any two coprime squarefree positive integers n, m. He proves that this set is a group, under the operation of taking the convolution product (produit intégral).

C. G. Lekkerkerker (Amsterdam)

17:

Stein, Sherman K. Unions of arithmetic sequences. Math. Ann. **134** (1958), 289–294.

Let J be the set of all integers, and for $d, b \in J$, $d > 0$, let $(d:b) = \{dx + b \mid x \in J\}$. The author proves some theorems about sets of the form $\mathbf{U}\,(d_i:b_i)$, among which is the following uniqueness theorem. If $\mathbf{U}_{i=1}^{n}(d_i:b_i) = \mathbf{U}_{j=1}^{m}(e_j:c_j)$, where the $(d_i:b_i)$ are pairwise disjoint and incongruent and so are the $(e_j:c_j)$, then $m=n$ and, with appropriate ordering, $(d_i:b_i) = (e_i:c_i)$. Thus he obtains a new proof of the theorem of Mirsky and Newman that J is not the union of a finite collection of disjoint incongruent sequences unless the collection consists simply of J itself. He also states three interesting unsolved problems.

N. J. Fine (Princeton, N.J.)

18:

Rossi, Francesco Saverio. Sul carattere di una particolare congruenza. Boll. Un. Mat. Ital. (3) **12** (1957), 694–697.

The congruence $(p-x)! \equiv \lambda \pmod p$ with $1 \leqq \lambda < p$, $x < p$, is considered and some classes of solutions are given. For example, if $x=3$ and p is an odd prime of the form $2\lambda+1$ then $(p-3)! \equiv \lambda \pmod p$. More generally, if p is a prime of the form $p = (h-1)\lambda + (-1)^{h-1}$, then $(p-h)! \equiv \lambda \pmod p$ and $(p-3)! \equiv \frac{1}{2}(h-1)!\lambda$ if h is odd, $(p-3)! \equiv \frac{1}{2}h!\lambda - 1$ if h is even. The case when p is composite is also discussed.

W. H. Simons (Vancouver, B.C.)

19:

Moessner, Alfred. Einige zahlentheoretische Untersuchungen und diophantische Probleme. Bul. Inst. Politech. Iaşi **4** (1949), 143–154.

Many systems of the Tarry-Escot type are discussed, and numerous numerical solutions are given. For example a non-trivial integer solution is given for the system

$$\sum_{i=1}^{20} E_i^n = \sum_{i=1}^{20} F_i^n, \quad 1 \leqq n \leqq 10.$$

W. H. Mills (New Haven, Conn.)

20:

Pignataro, Salvatore. Sulle terne pitagoriche. Ricerca, Napoli (2) **8** (1957), Gennaio-Giugno, 73–98.

The author has considered the integral right triangles whose legs are consecutive integers [Archimede **6** (1954), 185]. In the present paper he studies triangles whose legs differ by a constant. Sequences of such triangles arise out of the multiple solutions of the Pell equation and hence have sides satisfying second order linear recurrences. The treatment is in great detail.

D. H. Lehmer (Cambridge, England)

21:

Carlitz, L. Note on a paper of Dieudonné. Proc. Amer. Math. Soc. **9** (1958), 32–33.

Soit (a_n) une suite de nombres rationnels, et considérons la série formelle $\exp(\sum_{m=1}^{\infty} a_m x^m) = \sum_{n=0}^{\infty} c_n x^n$. L'auteur montre que pour que les c_n soient tous entiers, il faut et il suffit que, pour tout $k \geqq 1$, on ait $\sum_{r \mid k} r a_r \mu(s) \equiv 0 \pmod k$, où μ est la fonction de Möbius. Il donne une condition semblable pour que les c_n aient leurs dénominateurs produits de facteurs premiers dans un ensemble donné P de nombres premiers. Lorsque P se réduit à un seul élément p, il retrouve ainsi un résultat récent du rapporteur [Proc. Amer. Math. Soc. **8** (1957), 210–214; MR **19**, 290].

J. Dieudonné (Evanston, Ill.)

22:

Hua, Loo-keng. On exponential sums. Sci. Record (N.S.) **1** (1957), 1–4.

Let k be an integer $\geqq 2$ and let $(h, q)=1$. The exponential sum investigated in this paper is defined by $S_q=\sum_{x=1}^{q}\exp(2\pi i h x^k/q)$. The main theorem proved states that if $\varepsilon>0$ is fixed, then

$$\sum_{x=1}^{m}\exp(2\pi i h x^k/q)=\frac{m}{q}S_q+O(q^{1+\varepsilon}),$$

where the constant implied by the O symbol depends only on k and ε. This result improves substantially previous results by the author [J. Chinese Math. Soc. **2** (1940), 301–312; MR **2**, 347] and by I. M. Vinogradov [Selected works, Izdat. Akad. Nauk SSSR, Moscow, 1952; MR **14**, 610; pp. 291–295]. The author's principal tool is A. Weil's celebrated general theorem on exponential sums [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR **10**, 234], which may be formulated as follows: Let p be a prime and $f(x)=a_k x^k+\cdots+a_1 x+a_0$, $p\nmid(a_k, \cdots, a_1)$. Then

$$\left|\sum_{x=1}^{p}\exp(2\pi i f(x)/p)\right|\leqq k p^{\frac{1}{2}}.$$

A. L. Whiteman (Los Angeles, Calif.)

23:

Hua, Loo-Keng. On the major arcs of Waring problem. Sci. Record (N.S.) **1** (1957), no 3, 17–18.

Let $T(\alpha)$ denote the exponential sum $\sum_{x=1}^{P}\exp\{2\pi i x^k\alpha\}$ where $P=[N^{1/k}]$. In a previous note [# 22 above] the author had obtained the estimate

$$T(\alpha)=\frac{1}{q}\sum_{x=1}^{q}\exp\{2\pi i(h/q)x^k\}\int_0^P\exp\{2\pi i\beta y^k\}dy+O(q^{1+\varepsilon}),$$

on the major axes $\mathfrak{M}_{h,q}$ of Waring's problem. In the present note the author states that, as a result, the estimate

$$\sum_{\mathfrak{M}_{h,q}}\int_{\mathfrak{M}_{h,q}}T^s(\alpha)e^{-2\pi i N\alpha}d\alpha\sim\mathfrak{S}(N)\frac{\Gamma^s(1+1/k)}{\Gamma(s/k)}N^{(s/k)-1},$$

where $\mathfrak{S}(N)$ denotes the singular series, can be proved for $s\geqq k+1$ (instead of $s\geqq 2k+1$), which is best possible in a certain sense. *E. G. Strauss* (Los Angeles, Calif.)

24:

Mikolás, Miklós. A simple proof of the functional equation for the Riemann zeta-function and a formula of Hurwitz. Acta Sci. Math. Szeged **18** (1957), 261–263.

The functional equation of the Riemann zeta-function or, more generally, the Hurwitz formula for

$$\zeta(s,\mu)=\sum_{n=0}^{\infty}(\mu+n)^{-s}\quad(0<\mu\leqq 1,\ \Re(s)>1)$$

is deduced by expanding $\zeta(s,\mu)$ in a Fourier series $\sum_{-\infty}^{\infty}c_n e^{2\pi i n\mu}$ for $0<\Re(s)<1$, by means of the equation

$$\zeta(s,\mu)=\lim_{N\to\infty}\left[\sum_{n=0}^{N}(\mu+n)^{-s}-\frac{N^{1-s}}{1-s}\right]\quad(\Re(s)>0).$$

H. Kober (Birmingham)

25:

Franklin, J. N. On the equidistribution of pseudo-random numbers. Quart. Appl. Math. **16** (1958), 183–188.

For real x let $\{x\}$ denote the fractional part $x-[x]$. Many computer programs that use pseudo-random sequences generate them by means of the recurrence $x_n=\{Nx_{n-1}\}$ ($x_0\neq 0$), where N is an integer so chosen, in terms of the word size and number system of the computer, that the period of the sequence should be maximum. The author considers the slightly more general recurrence $x_n=\{Nx_{n-1}+\theta\}$, $0\leqq\theta<1$, in the somewhat theoretical case in which x_0 is irrational, a term not understood by computers. He applies the ergodic theorem to show that for almost all choices of x_0 the values of the x's are equidistributed in the unit interval. The case of $N=1$ is that of the classical paper of H. Weyl [Math. Ann. **77** (1916), 313–352]. If $\nu(n)$ denotes the number of x's among the first n that lie in a given interval, then for almost all x_0 the ratio $\nu(n)/n$ tends to the length of that interval as n increases. The argument uses a very general theorem of F. Riesz [Comment. Math. Helv. **17** (1945), 221–239; MR **7**, 255].

D. H. Lehmer (Cambridge, England)

26:

Mann, Henry B. On integral bases. Proc. Amer. Math. Soc. **9** (1958), 167–172.

Let F be a field of characteristic $\neq 2$ which is the quotient field of a Dedekind ring O. Let F' be a (separable) quadratic extension of F and O' the integral closure of O in F'. The author then proves that O' has an integral base over O if and only if F' is generated over F by an element α such that (α^2) is the discriminant of F'/F. Using this, he also proves that if O is not a principal ideal ring, there exists a quadratic extension F' of F for which O' has no integral base over O. *K. Iwasawa* (Cambridge, Mass.)

27:

Hanly, Virginia; and Mann, H. B. A note to the paper on integral bases by H. B. Mann. Proc. Amer. Math. Soc. **9** (1958), 173–174.

The result of the paper reviewed above is generalized to include the case of fields of characteristic 2.

K. Iwasawa (Cambridge, Mass.)

28:

Burgess, D. A. The distribution of quadratic residues and non-residues. Mathematika **4** (1957), 106–112.

This paper presents the first significant improvement on Vinogradov's estimate [Trans. Amer. Math. Soc. **29** (1927), 209–217] for the magnitude of the least positive quadratic non-residue d, modulo a prime $p>2$. The new result is $d=O(p^\alpha)$ as $p\to\infty$, for any fixed $\alpha>\frac{1}{4}e^{-1}$; the improvement being essentially a factor of $\frac{1}{4}$ in the exponent α. The theorem on which this is based is as follows:

Theorem 1. Let δ and ε be any fixed positive numbers and let $\left(\frac{n}{p}\right)$ denote the usual Legendre quadratic character. Then for all sufficiently large p and N, we have

$$\left|\sum_{n=N+1}^{N+H}\left(\frac{n}{p}\right)\right|<\varepsilon H,$$

provided that $H>p^{\frac{1}{4}+\delta}$.

This is an important variation on the classical inequality

$$\left|\sum_{n=N+1}^{N+H}\left(\frac{n}{p}\right)\right|<p^{\frac{1}{2}}\log p,$$

and its derivation is altogether deeper; being a consequence of A. Weil's theorem [Sur les courbes algébriques et les variétés qui s'en déduisent, Hermann, Paris, 1948; MR **10**, 262; Chaps. II, IV] on a generalized Riemann hypothesis for algebraic function-fields.

J. H. H. Chalk (Hamilton, Ont.)

29:

Lamprecht, Erich. Invariante Zetafunktionen arithmetischer Funktionenkörper. Abh. Math. Sem. Univ. Hamburg **22** (1958), 71–83.

Suppose that A is an algebraic function field of one variable whose precise coefficient field is a finite algebraic number field K. Denote by $B=\{\mathfrak{p}\}$ the set of all inequivalent rank one valuations of K. A prolongation P of $\mathfrak{p}$ to A is called a functional valuation of A provided there is an $x\in A-K$ such that (*) $v_P(\sum_{i=0}^{n}a_ix^i)=\mathrm{Min}_i(v_\mathfrak{p}(a_i))$ for every polynomial $\sum a_ix^i$ in $K[x]$. Furthermore, P is called regular if (K) $KP=K\mathfrak{p}$ is the precise field of constants of AP (residue class fields being denoted by postscripts), (U) P and $\mathfrak{p}$ have the same value group, and (G) A/K and $AP/K\mathfrak{p}$ have the same genus g. The author reformulates results on the existence of regular prolongations P for given $\mathfrak{p}\in B$, e.g., if $g\geqq 2$ then almost all $\mathfrak{p}$ have a uniquely determined regular P. Term then $\mathfrak{p}$ A-regular if it possesses some regular prolongation. Then it is recalled that almost all $\mathfrak{p}$ are A-regular, and, in particular, that the isomorphism types AP are unique. Denoting by $B_A(x)$ the subset of B consisting of $\mathfrak{p}$'s with prolongations P such that (*) holds for fixed x, the author defines the zeta function $Z_A(s,x)$ as $\prod_P Z_{AP}(s)$, $P\in B_A(x)$. Using properties of the congruence zeta functions $Z_{AP}(s)$ and of the zeta function $Z_K(s)$ of K it follows quickly that the above product converges for $K(s)>2$ and is in this half plane free of zeros and poles, furthermore that its poles and zeros in $\mathrm{R}(s)>3/2$ are those of $Z_K(s-1)$. In order to obtain a zeta function $Z_A'(s)$ which is invariantly attached to the field A/K the author uses the set B_A' of all A-regular prime divisors of K, picks for each $\mathfrak{p}$ a fixed regular prolongation P, and sets $Z_A'(s)=\prod_P Z_{AP}(s)$, P determined as indicated by all $\mathfrak{p}\in B_A'$. Since almost all $P\in B_A(x)$ are regular and inert over $K(x)$, it follows $Z_A'(s)$ and $Z_A(s,x)$ are related by a correction factor $\prod_P Z_{AP}(s)\prod_Q Z_{AQ}(s)^{-1}$, where P is regular and in $B_A(x)$, whereas Q is in $B_A(x)$ but not regular. This factor is regular for $\mathrm{R}(s)>1$ and has on $\mathrm{R}(s)=0,\ \frac{1}{2},\ 1$ only a finite number of inequivalent zeros and poles; furthermore $Z_A'(s)$ converges absolutely for $\mathrm{R}(s)>2$. The author raises the important question on the modifications which are necessary to obtain an invariant zeta function or A/K which satisfies a simple functional equation; finally, special cases are discussed which illuminate the serious problems arising in this connection.

O. F. G. Schilling (Chicago, Ill.)

30:

Armitage, J. V. Euclid's algorithm in certain algebraic function fields. Proc. London Math. Soc. (3) **7** (1957), 498–509.

Let Ω denote the field $GF[p]$ with p elements, p an odd prime, let $\Omega[t]$ denote the integral domain of polynomials in an indeterminate t with coefficients in Ω, and let $\Omega(t)$ denote the quotient field of $\Omega[t]$. For any element f of $\Omega[t]$ define $|f|=p^n$, where n is the degree of f, extend this valuation to $\Omega(t)$ in the obvious way, and let $\Omega\{t\}$ denote the perfect extension of $\Omega(t)$ with respect to $|\cdot|$, so that $\Omega\{t\}$ consists of all formal Laurent series $a_nt^n+a_{n-1}t^{n-1}+\cdots+a_0+a_{-1}t^{-1}+\cdots$, $a_i\in\Omega$, $a_n\neq 0$. Artin [Math. Z. **19** (1924), 153–246] pointed out the analogy to ordinary number theory with $\Omega[t]$, $\Omega(t)$ and $\Omega\{t\}$ corresponding, respectively, to the domain of rational integers, the rational field and the real field. He also studied quadratic extensions Q of $\Omega(t)$, and called Q "real" if Q is isomorphic to a subfield of $\Omega\{t\}$, for which a necessary and sufficient condition is that $Q\approx\Omega(t,\sqrt{d})$, $d=d_nt^n+d_{n-1}t^{n-1}+\cdots+d_1\in\Omega$, n even and d_n a quadratic residue modulo p. The author of the present paper, which is part of his London dissertation (1956), finds necessary and sufficient conditions for certain algebraic extensions K of $\Omega(t)$ to be euclidean, meaning that for any non-integral element α of K there exists an integral element ξ of K such that $|N(\xi-\alpha)|<1$, where $N(\cdot)$ is the usual norm. He gives incomplete results for K cubic, but completely disposes of real quadratic extensions Q with the theorem that Q is euclidean if and only if its discriminant D has the value $|D|=p^2$. His methods are similar to those of Davenport [Quart. J. Math. Oxford Ser. (2), **1** (1950), 54–62; Acta Math. **84** (1950), 159–179; MR **11**, 582; **12**, 594]. His proofs are based on a particular case of a theorem of Mahler on the product of n linear forms in n variables with coefficients in a field of series [Ann. of Math. (2) **42** (1941), 488–522; MR **2**, 350].

R. Hull (Pacific Palisades, Calif.)

31:

Klingen, Helmut. Zur Theorie der hermitischen Modulfunktionen. Math. Ann. **134** (1958), 355–384.

Let Σ be an imaginary quadratic number field of discriminant $\delta\neq -4,\ -3$. Generalizing Siegel's modular functions of degree n [Math. Annalen **116** (1939), 617–657; MR **1**, 203], one makes the following definitions: Let E be the unit matrix of order n, set $I=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix}$ and, for every matrix Z, with elements in Σ, denote by $\tilde{Z}$ the transposed complex conjugate $\bar{Z}'$; define $\mathfrak{H}_n$, the "generalized upper half-plane" (of n^2 complex dimensions) as the set of matrices Z, such that the hermitian matrix $(Z-\tilde{Z})/2i$ is positively definite. The set $\mathfrak{M}_n$ of square matrices M of order $2n$, with elements in Σ and satisfying $\tilde{M}IM=I$ constitutes the hermitian modular group. If $Z\in\mathfrak{H}_n$ and $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathfrak{M}_n$, then also $M\langle Z\rangle=(AZ+B)(CZ+D)^{-1}\in\mathfrak{H}_n$. Meromorphic functions of n^2 complex variables, defined on $\mathfrak{H}_n$ and invariant under the transformations $Z\to M\langle Z\rangle$, $M\in\mathfrak{M}_n$, are called hermitian modular functions. For $n=1$, one obtains the classical modular functions. One knows [H. Braun, Ann. of Math. (2) **50** (1949), 827–855; **53** (1951), 143–160; MR **11**, 333; **12**, 482] how to determine a fundamental domain of $\mathfrak{M}_n$ in $\mathfrak{H}_n$, and that the generalized Eisenstein series $\varphi_g(Z)\equiv\sum_{(C,D)}|CZ+D|^g$ (summation extended over the coprime matrices C,D of Σ) converge for $g<-2n$. The author proves that $\varphi_g(Z)$ are holomorphic modular forms of dimension g in $\mathfrak{H}_n$. Hermitian modular functions are now defined as quotients of modular forms of equal dimensions. They form a field of degree of transcendency at least n^2 over the complex numbers. The most striking difference between the present theory and that of Siegel's modular functions of degree n is that in the hermitian case one has to take into account the congruence subgroups already for the proof of the fundamental (basis) theorem on functions modular under the full hermitian group $\mathfrak{M}_n$ [see also H. Braun, Abh. Math. Sem. Univ. Hamburg **19** (1955), 134–148; MR **16**, 801]. The author studies the fields of functions modular under subgroups of $\mathfrak{M}_n$. Restricting $\mathfrak{H}_n$ by the condition $Z=Z'$, one obtains modular functions, forming a subfield of degree $\frac{1}{2}n(n+1)$ of the field Λ of (Siegel) modular functions of degree n. Finally, the author discusses the characterization of hermitian modular forms by differential equations. The Eisenstein series corresponding to certain subgroups of $\mathfrak{M}_n$ satisfy a set of $2n^2$ differential equations, which characterize pencils of

automorphic forms. The paper concludes with the establishment of functional relations between different pencils of hermitian modular forms.

E. Grosswald (Philadelphia, Pa.)

32:

Ridout, D. Rational approximations to algebraic numbers. Mathematika **4** (1957), 125–131.

If α is any real algebraic number and if $k>2$, then the Thue-Siegel-Roth theorem asserts that there are at most finitely many rational numbers p/q ($q>0$) satisfying the inequality $|\alpha-p/q|<q^{-k}$. The author shows that the lower bound for k can be reduced if certain conditions are imposed on p and q. His result reads:

Let α be any algebraic number other than 0; let $P_1, \cdots, P_s, Q_1, \cdots, Q_t$ be distinct primes and let μ, ν, c be real numbers satisfying $0\leqq\mu\leqq 1$, $0\leqq\nu\leqq 1$, $c>0$. Let p, q be restricted to integers of the form $p=p^*P_1^{\rho_1}\cdots P_s^{\rho_s}$, $q=q^*Q_1^{\sigma_1}\cdots Q_t^{\sigma_t}$, where $\rho_1, \cdots, \rho_s, \sigma_1, \cdots, \sigma_t$ are non-negative integers and p^*, q^* are integers satisfying $0<|p^*|\leqq cp^{\mu}$, $0\leqq|q^*|\leqq cq^{\nu}$. Then, if $k>\mu+\nu$, there are at most a finite number of solutions of the inequality $0<|\alpha-p/q|<q^{-k}$.

There is a similar, but weaker, result in Th. Schneider's new book on transcendental numbers, "Einführung in die transzendenten Zahlen", Springer, Berlin-Göttingen-Heidelberg, 1957 [MR **19**, 352], p. 13, Satz 6.

J. Popken (Amsterdam)

33:

Mahler, K. On the fractional parts of the powers of a rational number. II. Mathematika **4** (1957), 122–124.

The author shows: Let α be any positive algebraic number; let u and v be relatively prime integers satisfying $u>v\geqq 2$ and let $\varepsilon>0$. Then the inequality

$$|\alpha(u/v)^n-\text{nearest integer}|<\varepsilon^{-\varepsilon n}$$

is satisfied by at most a finite number of positive integers n.

The proof uses Ridout's extension of the Thue-Siegel-Roth theorem [see the preceding review].

The special case $\alpha=1$, $u=3$, $v=2$, $\varepsilon=\log\frac{4}{3}$ is of interest for the solution of Waring's Problem. From this and from well-known results about $g(k)$ the author deduces: Except possibly for a finite number of values of k, we have

$$g(k)=2^k+[(3/2)^k]-2.$$

J. Popken (Amsterdam)

34:

Sós, Vera T. On the theory of diophantine approximations. I. Acta Math. Acad. Sci. Hungar. **8** (1957), 461–472.

Denote by $\{\alpha\}$ the fractional part of the real number α. For real α and positive integer N put

$$C_\alpha(N)=-\frac{N}{2}+\sum_{1\leqq n\leqq N}\{n\alpha\}.$$

It was shown by A. Ostrowski [Abh. Math. Sem. Univ. Hamburg **1** (1921), 77–98] that $C_\alpha(N)$ is unbounded for every irrational α. In this paper the author answers a question raised by Ostrowski by showing that there exist real α such that $C_\alpha(N)$ is bounded below (or above). It is remarked that a modification of the proof would produce an α such that $C_\alpha(N)>-\varepsilon$ for all N, where ε is an arbitrarily small positive number: but that it is impossible for $C_\alpha(N)$ to be positive for every N.

The proof depends on an explicit formula for $C_\alpha(N)$ related to one given by Ostrowski, but in which in addition to the denominators of the convergents to α (the Hauptnenner), as in Ostrowski, the bye-denominators (Nebennenner) also occur.

The formula is motivated by a 'geometric' interpretation of the continued fraction algorithm, which is stated to generalize naturally to an algorithm appropriate to inhomogeneous problems.

J. W. S. Cassels (Cambridge, England)

35:

Schönhage, Arnold. Zum Schubfächerprinzip im linearen Intervall. Arch. Math. **8** (1957), 327–329.

Consider the sequences $A=(a_0, a_1, \cdots)$ with $a_0=1$, $a_1=0$, $0\leqq a_\nu\leqq 1$ and $a_\mu\neq a_\nu$ for $\mu\neq\nu$. The points $a_0, a_1, \cdots, a_n$ of such a sequence divide the interval $0\leqq x\leqq 1$ into n subintervals; let u_n be the shortest subinterval, so that $0<u_n\leqq 1/n$, and let $\omega=\sup_A\{\liminf nu_n\}$. A. Ostrowski showed [Arch. Math. **8** (1957), 1–10; MR **19**, 638] that $\frac{1}{2}\leqq\omega\leqq 1/\log 4=0.7213\cdots$. It is shown here that $\omega=1/\log 4$. It is pointed out in a footnote that this result has been obtained earlier by Erdös and de Bruijn [Proc. Nederl. Akad. Wetensch. Proc. Ser. A. **52**, 46–49 (1949); MR **11**, 423].

W. J. LeVeque (Göttingen)

36:

Ostrowski, Alexander. Bemerkungen zu meiner Mitteilung: Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall. Arch. Math. **8** (1957), 330.

Acknowledgment of the priority of Erdös and de Bruijn, as regards the evaluation of ω [see the preceding review]. Certain 'finite' results of the paper of the title [and also Jber. Deutsch. Math. Verein. **60** (1957), 33–39; MR **19**, 638] were not obtained by those authors.

W. J. LeVeque (Göttingen)

37:

Toulmin, G. H. Subdivision of an interval by a sequence of points. Arch. Math. **8** (1957), 158–161.

Another proof that $\omega=1/\log 4$. [See the two preceding reviews.] It is also shown here that the bound is actually attained, for example by the sequence in which $a_n=\{\log_2(2n-1)\}$ for $n\geqq 1$, where $\{x\}=x-[x]$ and $\log_2$ denotes the logarithm to base 2. Furthermore, it is shown that $\limsup nv_n\geqq 1/\log 2$ for all sequences A, where v_n is the longest of the subintervals into which the points $a_0, a_1, \cdots, a_n$ divide the unit interval. This lower bound is also attained by the above sequence.

W. J. LeVeque (Göttingen)

38:

Armitage, J. V. The product of n linear forms in a field of series. Mathematika **4** (1957), 132–137.

Let K be a field, t an indeterminate, $K\{t\}$ the field of all formal series $\phi=\sum_{k=-\infty}^{n}a_kt^k$ where $a_k\in K$, and let $|\phi|$ denote the valuation on $K\{t\}$ defined by $|0|=0$, $|\phi|=e^m$ if $a_m\neq 0$. Let $L_i=\sum_{j=1}^{n}\alpha_{ij}u_j$ $(i=1, 2, \cdots, n)$, where $\alpha_{ij}\in K\{t\}$, be n linear forms of determinant $D\neq 0$. By K. Mahler [Ann. of Math. (2), **42** (1941), 488–522; MR **2**, 350], there exist polynomials $u_1, \cdots, u_n\in K[t]$ not all zero such that

$$|L_1L_2\cdots L_n|\leqq e^{-(n-1)}|D|.$$

In the opposite direction, the author proves that this result is best-possible if K has the characteristic 0, or if K is a finite field of at least $n-1$ elements. If, however, K is a finite field of $q\leqq n-2$ elements, then the stronger inequality

$$|L_1L_2\cdots L_n|\leqq e^{-n+2-\gamma}|D|$$

can be satisfied; here γ is the integer defined by $\gamma\geqq(n-1)/q>\gamma-1$.

K. Mahler (Manchester)

39:

Woods, A. C. On a theorem of Minkowski's. Proc. Amer. Math. Soc. **9** (1958), 354–355.

Let K be a closed strictly convex body in R_n, symmetric about the origin O and let Λ be an arbitrary lattice in R_n. The successive minima $\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda)$ of K are defined as the least upper bounds of numbers $c_1, c_2, \cdots, c_n$, respectively, which have the property that c_iK contains at most $i-1$ linearly independent points of Λ within its interior $(i=1, 2, \cdots, n)$. Let $X_1, X_2, \cdots, X_n$ denote n linearly independent points of Λ such that $\mu_i(\Lambda)K$ contains $X_1, X_2, \cdots, X_i$ for $i=1, 2, \cdots, n$, and let $Z_1, Z_2, \cdots, Z_n$ denote a basis for Λ satisfying

$$X_i = \sum_{j=1}^{i} g_{ij}Z_j \quad (i=1, 2, \cdots, n),$$

where the coefficients g_{ij} are integers with $g_{ii}>0$.

Denote by $X_{i1}, X_{i2}, \cdots, X_{ir_i}$ all the points of $\Lambda \cap \mu_i(\Lambda)K$ of the form

$$X_{is} = \sum_{j=1}^{i} g_{ij}^{(s)}Z_j,$$

where the coefficients $g_{ij}^{(s)}$ are integers with $g_{ii}^{(s)}>0$. Then the main result of the note is the inequality

$$\sum_{i=1}^{n} r_i \leqq 2^n - 1.$$

This clearly implies that if Λ is admissible for K, then there are at most 2^n-1 pairs of points $\pm X$ of Λ lying on the boundary of K, which is a familiar theorem of Minkowski. *J. H. H. Chalk* (Hamilton, Ont.)

40:

Lawton, B. Bounded representations of the positive values of an indefinite quadratic form. Proc. Cambridge Philos. Soc. **54** (1958), 14–17.

Theorem. If

$$f(x_1, \cdots, x_n) = \sum_i \sum_j f_{ij}x_ix_j$$

is a proper indefinite quadratic form with integral coefficients in the n variables $x_1, \cdots, x_n$ and $F=\max |f_{ij}|$, then $f(x)>0$ is always soluble in integers $x_1, \cdots, x_n$ satisfying

$$\max_i |x_i| < c_n F^{n/2},$$

where c_n depends only on n.

A counter-example shows that the exponent $n/2$ cannot be improved in general, and a stronger result is indicated when f has more than one positive eigenvalue.

The hypothesis that the coefficients of f should be integers is not used in the extremely simple proof.

J. W. S. Cassels (Cambridge, England)

COMMUTATIVE RINGS AND ALGEBRAS

See also 38.

41:

Smith, R. J. A determinant in continuous rings. Pacific J. Math. **7** (1957), 1701–1709.

The author, extending the results of J. Dieudonné [Bull. Soc. Math. France **71** (1943), 27–45; MR **7**, 3], gives a definition of a determinant in continuous rings of von Neumann. However, his method actually provides a proof of the following fact: An invertible element of a complete rank ring (not discrete) is always a limit, by the rank-metric, of multiples of elements of class 2. (x is said to be of class 2 if $(1-x)^2=0$.)

This shows that the commutator group of the multiplicative group of all invertible elements is dense in the latter. So his definition of a determinant is vacuous.

I. G. Amemiya (Kingston, Ont.)

42:

Moisil, Gr. C. Sur un problème algébrique. I. Com. Acad. R. P. Romîne **7** (1957), 577–579. (Romanian. Russian and French summaries)

43:

Moisil, Gr. C. Sur un problème algébrique. II. Com. Acad. R. P. Romîne **7** (1957), 637–639. (Romanian. Russian and French summaries)

Soient deux formes quadratiques données, S et T dans les variables $Z, X_1, \cdots, X_n$; trouver les polynômes U et V de manière que le polynôme $P_0=US+VT$ soit indépendant de Z. *Du résumé de l'auteur*

44:

Manteuffel, Karl. Übersicht über die Irreduzibilitätsbeweise der Kreisteilungsgleichung. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **1** (1957), 69–75.

The author gives a survey, classification and brief discussion of the various proofs for the irreducibility of the cyclotomic polynomials. A bibliography is appended. *F. A. Behrend* (Melbourne)

ALGEBRAIC GEOMETRY

45:

d'Orgeval, Bernard. A propos des variétés à n dimensions, possédant une congruence de variétés de dimension inférieure. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 146–154.

Detta parapicardiana una varietà algebrica avente tutti i sistemi canonici d'ordine zero (varietà che se fosse esatta una congettura di Severi sarebbe una varietà picardiana), l'Autore estende alle varietà algebriche che posseggono una congruenza di varietà parapicardiane alcuni dei risultati stabiliti da L. Roth per le varietà che posseggono una congruenza di varietà di Picard.

D. Gallarati (Genova)

46:

***Wallace, Andrew H. Homology theory on algebraic varieties.** International Series of Monographs on Pure and Applied Mathematics. Vol. 6. Pergamon Press, New York-London-Paris-Los Angeles, 1958. viii+115 pp. $5.50.

The object of this monograph is to give rigorous proofs of three main theorems of Lefschetz formulated in "L'analysis situs et la géométrie algébrique" [Gauthier-Villars, Paris, 1924]. These theorems may be briefly described as follows. Let V be a non-singular r-dimensional algebraic variety in complex projective space, and let V_0 be a non-singular hyperplane section of V. Then Lefschetz's first main theorem states that all cycles of dimension less than r on V are homologous to cycles on V_0. Now V_0 may be taken as a member of a pencil of hyperplane sections of V, a pencil which contains only a

finite number of singular sections. Lefschetz's second main theorem, interpreted in terms of relative homology, shows how to obtain a set of generators for $H_r(V, V_0)$, one of which is associated in a certain way with each of these singular sections. The third main result of Lefschetz concerns the Poincaré formula, which describes the variation of cycles of V_0 as this section is made to vary within a pencil of sections.

The main technique used in the proofs is the fibring of the variety V by a pencil Π of hyperplane sections, the fibring being singular in two respects; namely all the fibres meet along the axis P of Π, and a finite number of them carry singularities. This technique, which has appeared together with the proof of the first main theorem in a recent paper of the author [Ann. of Math. (2) **63** (1956), 248–271; MR **17**, 1131], is given in a modified form in the first four chapters of this monograph. The proofs of the second and third main theorems are given in chapters V–VII using the same technique.

Sze-tsen Hu (Detroit, Mich.)

47:

Murre, J. P. Intersection multiplicities of maximal connected bunches. Amer. J. Math. **80** (1958), 311–330.

Following a suggestion of André Weil, a theory of intersection-multiplicities is developed for maximal connected bunches in the intersection of two varieties of complementary dimension on a complete ambient variety, subject to certain restrictions on the ambient variety. It is assumed that the ambient variety is projective, and that every point of the bunch considered is simple on the ambient variety. When the components of intersection are proper, the theory has already been developed (for general characteristic p) by Weil, and the proofs in the paper under discussion proceed by induction on a suitably defined integer, called the 'excess' of the bunch, this excess being zero in the proper case.

It is refreshing to see some of the more geometrical methods developed by Severi, van der Waerden, Hodge and the reviewer being combined with the more abstract methods of Weil and Zariski. A synthesis of these methods is long overdue. *D. Pedoe* (Khartoum)

48:

Murre, J. P. On a connectedness theorem for a birational transformation at a simple point. Amer. J. Math. **80** (1958), 3–15.

Another proof is given of the theorem that the total transform of a simple point by a birational transformation is connected. In fact, it is proved that the set of points in the total transform is 'linearly' connected, in the sense that every pair of points can be connected by a sequence of rational curves in the set.

Some applications are derived to specialisations of a complete set of conjugates over a purely transcendental function field. *D. Pedoe* (Khartoum)

LINEAR ALGEBRA

49:

Herstein, I. N. On a theorem of Putnam and Wintner. Proc. Amer. Math. Soc. **9** (1958), 363–364.

It is shown that if a and b are regular $n\times n$ matrices over a field of characteristic $p>n$ and if a commutes with $ab-ba$, then $ab^{-1}a^{-1}b-1$ is nilpotent.

C. R. Putnam (Lafayette, Ind.)

50:

Zelmer, Violeta. Sur la construction d'une matrice orthogonale et symétrique. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 157–159. (Romanian. Russian and French summaries)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 67.

51:

Szász, F. Les anneaux ne contenant que des sous-anneaux propres cycliques. Czechoslovak Math. J. **7(82)** (1957), 21–25. (Russian summary)

On appelle anneau cyclique un anneau associatif dont le groupe additif est cyclique. Dans l'article présent, l'auteur considère les anneaux, R, caractérisés par la propriété que tous les sous-anneaux propres d'un anneau R sont cycliques. Le résultat consiste en ceci, que les anneaux R sont précisément les suivants: les anneaux cycliques, l'anneau zéro construit au-dessus du groupe additif quasi-cyclique du type p^∞, les corps finis de l'ordre p et p^q (p, q étant des nombres premiers) et les anneaux de l'ordre p^2. On a encore le corollaire suivant: Un groupe abélien dont tous les sous-groupes propres sont cycliques est ou bien cyclique lui même ou bien possède l'ordre p^2 ou bien, finalement, est du type p^∞.

O. Borůvka (Brno)

52:

Divinsky, Nathan. D-regularity. Proc. Amer. Math. Soc. **9** (1958), 62–71.

An element r of a ring R is called right D-regular if $r=rs$ for some $s\in R$. The union of all right D-regular right ideals of R is an ideal A of R. A study is made of what it means for $A=0$ and $A=R$ if R satisfies certain chain conditions. *R. E. Johnson* (Northampton, Mass.)

53:

Ponizovskiĭ, I. S. On the determinant of matrices with elements from a certain ring. Mat. Sb. N.S. **45(87)** (1958), 3–16. (Russian)

Let R be a ring with minimum condition and unit-element e satisfying $2e\neq 0$, and denote by R_n^* the multiplicative semigroup of the ring R_n of $n\times n$ matrices over R. Given a commutative semigroup S with unit-element ε, the author defines a determinant function on R_n with values in S as a homomorphism $\Delta: R_{n*}\to S$ such that $\Delta(E)=\varepsilon$ (where E is the unit-element of R_n). If $M\in R_n$, he calls $\Delta(M)$ the determinant of M. He shows that (i) multiplying the elements of the ith row on the left by $b\in R$ multiplies $\Delta(M)$ by a factor $\Delta(b)$ independent of i, (ii) adding a left-multiple of one row to another leaves $\Delta(M)$ unchanged, (iii) interchanging two rows multiplies $\Delta(M)$ by $\Delta(-e)$. The statements obtained by replacing 'row' by 'column' and 'left' by 'right' also hold, but not if only 'rows' are replaced by 'columns', as simple examples show. — If in particular the elements of R are represented by $d\times d$ matrices over a field F, $a\to D(a)$ ($a\in R$, $D(e)=I$), then the elements of R_n may be represented by $nd\times nd$ matrices, $\varphi_D: M\to M_D$ say, and $\Delta(M)=\det M_D$ defines a determinant function on R_n. The author notes that in this case M is invertible if and only if $\Delta(M)\neq 0$; and if further, R is an algebra of finite dimension over F, he proves that the minimal number of generators of the group of values $\Delta(M)$ (M invertible) equals the number of inequivalent irreducible representations of R in F. *P. M. Cohn* (Manchester)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

54:

Kokoris, Louis A. Some nodal noncommutative Jordan algebras. Proc. Amer. Math. Soc. **9** (1958), 164–166.

A nodal algebra A over a field F is a power-associative algebra with 1 such that every element has the form $\alpha 1+z$, $\alpha \in F$, z nilpotent, while A is not of the form $A=F1+N$ for a nil subalgebra (ideal) N of A. It is known that nodal algebras do not exist among the following classes of algebras: alternative (including associative) algebras, commutative Jordan algebras of characteristic $\neq 2$, noncommutative Jordan algebras of characteristic 0.

The author constructs nodal noncommutative Jordan algebras of characteristic $p>2$ as follows: let m and n be any two positive integers, and consider the p^{m+n}-dimensional commutative associative algebra $B=F[x, y]$, $x^{p^m}=y^{p^n}=0$, in which multiplication is denoted by $u \cdot v$. Let g be any nilpotent element of B. Let A be an algebra which is the same vector space as B, but in which multiplication is defined by bilinearity and

$$(x^a \cdot y^b)(x^c \cdot y^d)=(ad-bc)\,(x^{a+c-1} \cdot y^{b+d-1}-x^{a+c} \cdot y^{b+d} + x^{a+c-1} \cdot y^{b+d-1} \cdot g)+x^{a+c} \cdot y^{b+d}$$

(where the following conventions are made: $x^0=y^0=1$; $x^a \cdot y^b=0$ whenever at least one exponent is negative). Then $u \cdot v=\frac{1}{2}(uv+vu)$, so that $B=A^+$, and A is a nodal noncommutative Jordan algebra. The author proves that A is simple if and only if $m=n=1$.

An example of a simple nodal noncommutative Jordan algebra of characteristic 2 is also given in the paper.

R. D. Schafer (Princeton, N.J.)

55:

***Jacobson, N. Jordan algebras.** Report of a conference on linear algebras, June, 1956, pp. 12–19. National Academy of Sciences — National Research Council, Washington, Publ. 502, v+60 pp. (1957).

This paper contains a survey of recent results in the foundations, the structure and representation theory, and the cohomology theory of Jordan algebras. In addition some general results concerning identities in the exceptional Jordan algebra are obtained and, specifically, it is shown that the two identities

$$\{\{axa\}x\{axb\}\}=\{\{\{axa\}xb\}xa\},$$
$$\{\{xbx\}a\{xbx\}\}=\{x\{b\{xax\}b\}x\},$$

where $\{abc\}=(ab)c+(bc)a-(ac)b$, hold in the exceptional Jordan algebra.

L. A. Kokoris (Chicago, Ill.)

56:

***Kleinfeld, Erwin. On alternative and right alternative rings.** Report of a conference on linear algebras, June, 1956, pp. 20–23. National Academy of Sciences—National Research Council, Washington, Publ. 502, v+60 pp. (1957).

The important results in the theory of alternative and right alternative rings are summarized and references to papers containing the proofs of these results are given. There is an extensive bibliography. The sketch of the proof of the fact that a simple, alternative nil ring of characteristic different from 2 and 3 is associative, is included. At the end of the paper there is a discussion of some unsolved problems.

L. A. Kokoris (Chicago, Ill.)

57:

Kleinfeld, Erwin. A note on Moufang-Lie rings. Proc. Amer. Math. Soc. **9** (1958), 72–74.

Malcev [Mat. Sb. N.S. **36(78)** (1955), 569–578; MR **16**, 997] has defined a ring to be Moufang-Lie in case it satisfies the anti-commutative law (i) $x^2=0$ and the identity (ii) $(xy, y, z)=y(x, y, z)$, for all elements x, y, z of the ring, where the Jacobian (x, y, z) is defined by $(x, y, z)=(xy)z+(yz)x+(zx)y$.

If, as the author points out, from a ring S we form a new ring R by replacing the product operation of S by the commutator-product $(x, y)=xy-yx$, then R is Lie or Moufang-Lie according as S is associative or merely alternative. In the latter case, the Jacobian of R is six times the associator of S. One would hope, then, that the Jacobian of any Moufang-Lie ring R satisfies identities analogous to those known for the associator of any alternative ring. The author establishes a number of such identities for R of characteristic not two. He uses them to prove the following theorem: If a Moufang-Lie ring R, of characteristic different from two, contains an element a such that $aR=R$, then R is a Lie ring.

The author adds the following remark: If an alternative ring S contains an element a such that $(a, S)=S$, then (without restriction as to characteristic), S is associative. He then shows his results are nonvacuous by producing an associative ring S with an element a such that $(a, S)=S$, namely the free associative ring with identity element, on two free generators a, b, reduced modulo the relation $ab-ba=1$.

R. H. Bruck (Madison, Wis.)

58:

Evans, Trevor. Nonassociative number theory. Amer. Math. Monthly **64** (1957), 299–309.

The logarithmetic of an algebra which does not satisfy any identities bears a close resemblance to ordinary arithmetic, and Etherington [Proc. Roy. Soc. Edinburgh. Sect. A. **62** (1949), 442–453; MR **10**, 677] has shown that its elements can be defined in terms of partitioned classes in analogy with the Frege-Russell definition of the natural numbers as classes of classes. As another approach, the author characterizes this logarithmetic, denoted N, by postulates analogous to the Peano postulates for the natural numbers, and develops its basic properties in a manner parallel to the usual development of the natural numbers. Results obtained include the analogues of the fundamental theorem of arithmetic and of Fermat's last theorem.

N is then extended to a system I in which subtraction is always possible, I being a left neoring [Bruck, Proc. Amer. Math. Soc. **6** (1955), 50–58; MR **16**, 1083]. Number theory is here complicated by the presence of many "units", forming multiplicatively an infinite cyclic group, and by the presence of many more primes than just the primes of N and their "associates", so that the analogue of the fundamental theorem of arithmetic has to be proved anew. Finally the concept of congruence is introduced, and homomorphic images of I are obtained which are the nonassociative analogues of finite arithmetics.

The paper concludes with suggestions for further work. {It would seem that the elements of I provide what is required for the exponents of positive and negative powers in loops and nonassociative division algebras, as the elements of N do for positive powers in groupoids and nonassociative algebras generally.}

I. M. H. Etherington (Edinburgh)

59:

Rédei, Ladislaus. Die einstufig nichtkommutativen endlichen Ringe. Acta Math. Acad. Sci. Hungar. **8** (1957), 401–442.

An EN-ring (einstufig nichtkommutativen Ringe) is a

noncommutative ring in which each subring is commutative. There are three immediate consequences of the definition of a finite EN-ring R: (i) any homomorphic image of R is either commutative or an EN-ring; (ii) R is generated by two noncommutative elements x and y; (iii) the additive group R^+ of R is a p-group. Using these the author obtains a complete classification and characterization of finite EN-rings. A nilpotent finite EN-ring is either isomorphic with or a homomorphic image of an EN-ring defined by the relations $p^m x=p^n y=x^r=y^s=0$, $x^2y=xyx=yx^2$, $xy^2=yxy=y^2x$, and $pyx=pxy$, with the natural numbers m, n, r and s satisfying the inequalities $1\leqq m\leqq n$, $r\geqq 2$, $s\geqq 2$, and $r\leqq s$ if $m=n$. Conversely these relations always define a nilpotent finite EN-ring. The other two classes of finite EN-rings, those which contain one-sided zero-divisors and those which do not contain one-sided zero-divisors, are similarly characterized. (The analogous problem for groups has also been studied by the author [Publ. Math. Debrecen **4** (1955–56), 303–324; MR **18**, 12].) *W. E. Deskins* (East Lansing, Mich.)

GROUPS AND GENERALIZATIONS

60:

Schenkman, Eugene. The similarity between the properties of ideals in commutative rings and the properties of normal subgroups of groups. Proc. Amer. Math. Soc. **9** (1958), 375–381.

Let A, B, and C be normal subgroups of a group G that satisfies the ascending chain condition for normal subgroups. The "similarity" follows from the following identifications: the sum of ideals corresponds to the product of normal subgroups; the product of ideals corresponds to the commutator of normal subgroups; the residual quotient of A by B is defined as $A:B=\{x\in G|xbx^{-1}b^{-1}\in A \text{ for all } b \text{ in } B\}$. $A:B$ is a normal subgroup of G that contains A, and $(\bigcap A_i):B=\bigcap(A:B_i)$, $A:(B_1B_2\cdots B_n)=A:B_1\cap\cdots\cap A:B_n$, and $A:B=A:AB$. C is prime in G if $[A,B]\subseteq C$ implies that either $A\subseteq C$ or $B\subseteq C$. If A is contained in the set union of the primes A_1, A_2, $\cdots$, A_n, then A is contained in one of them. A is radical if $B\subseteq A$ whenever $B^{(n)}\subseteq A$ for some n, where $B^{(n)}$ is the nth commutator group of B. Every radical of G has a unique minimal decomposition as an intersection of primes. A and B are complementary if $G=AB$. If $G=G^{(1)}$ and $A=A^{(1)}$, then A is the intersection of pairwise complementary subgroups C_i such that $C_i=C_i^{(1)}$ and such that the C_i are not intersections of pairwise complementary subgroups. *P. F. Conrad* (New Orleans, La.)

61:

Neumann, Hanna. On the intersection of finitely generated free groups. Addendum. Publ. Math. Debrecen **5** (1957), 128.

The author's improvement, given under restrictive assumptions in same Publ. **4** (1956), 186–189 [MR **18**, 11], of A. G. Howson's bound for the rank of the intersection of two finitely generated subgroups of a free group, holds without restriction according to a modification of the proof suggested by R. Baer.

D. G. Higman (Ann Arbor, Mich.)

62:

Kasch, F.; Kneser, M.; und Kupisch, H. Unzerlegbare modulare Darstellungen endlicher Gruppen mit zyklischer p-Sylow-Gruppe. Arch. Math. **8** (1957), 320–321.

Let p be a prime, G a group of order $g=p^sn$ with $(p,n)=1$, and P a p-Sylow subgroup of G. If $R=K(G)$ is the group ring of G over a field K of characteristic p, and if an R-module M is a direct sum $M=\sum^{\oplus}N_k$ of indecomposable S-modules N_k for $S=K(P)$, then M is R-isomorphic with an R-module direct summand of $R\otimes_S N_k$ for some k [D. G. Higman, Duke Math. J. **21** (1954), 377–381; MR **16**, 794]. If P is cyclic, there are precisely p^s non-isomorphic indecomposable S-modules, each S-isomorphic with one of the p^s ideals $\neq 0$ of S of dimensions 1, 2, $\cdots$, p^s. The authors point out that, consequently, in case P is cyclic, each indecomposable R-module is R-isomorphic with an indecomposable left ideal of R, and hence the number of non-isomorphic ones is $\leqq g$. They prove that the number is g if and only if P is a normal subgroup of G with abelian factor group G/P, and K contains a primitive mth root of unity, where m is the exponent of G/P. *D. G. Higman* (Ann Arbor, Mich.)

63:

Zacher, Giovanni. Sui gruppi finiti per cui il reticolo dei sottogruppi di composizione è distributivo. Rend. Sem. Mat. Univ. Padova **27** (1957), 75–79.

It is shown that the lattice of composition subgroups of a finite group is distributive if and only if the factors of prime power order of the normal chains of G are cyclic. This is to be compared with the result of G. Zappa for solvable groups [Boll. Un. Mat. Ital. (3) **11** (1956), 315–318; MR **18**, 280]. *D. G. Higman* (Ann Arbor, Mich.)

64:

Curzio, Mario. Sul reticolo dei sottogruppi di composizione di alcuni gruppi finiti. Boll. Un. Mat. Ital. (3) **12** (1957), 284–289.

The class of finite solvable groups such that the lattice of composition subgroups is a chain is characterized as consisting of the following two classes: (1) the cyclic groups of prime order, and (2) the groups of order $p^\alpha q^\beta$, $p>q$ primes, having cyclic Sylow subgroups, and such that the unique p-Sylow subgroup coincides with its centralizer. It is also shown that if the lattice of composition subgroups of a finite solvable group G is isomorphic with that of a non-cyclic group G^1 of prime power order, then G and G^1 have the same order and are lattice-isomorphic. *D. G. Higman* (Ann Arbor, Mich.)

65:

Feit, Walter. On the structure of Frobenius groups. Canad. J. Math. **9** (1957), 587–596.

A group of order hm is called a Frobenius group of type (h,m) if it contains a subgroup H of order h which is its own normalizer and has only the identity in common with each of its conjugates. These are the groups of order hm having faithful representations as transitive groups of permutations on m letters such that no permutation but the identity leaves two letters fixed, and at least one leaves exactly one letter fixed. By a theorem of Frobenius, such a group G contains a normal subgroup M of order m, consisting of all elements of G of order dividing m. Burnside proved that for even h, M is abelian. It has been conjectured that M is always nilpotent. The author proves that this is the case whenever M satisfies the condition (*): no subgroup of M has a composition factor which is ex-

ceptional. Here, a group K is called exceptional if it is a non-cyclic simple group for which the normalizer in K of each characteristic subgroup $\neq 1$ of each Sylow subgroup P is P. No exceptional groups are known. The author notes that (1) every solvable group and every group all of whose Sylow subgroups are abelian satisfies (*), and (2) if H is an automorphism group of order h of a group M of order m, such that no automorphism $\neq 1$ in H leaves any element $\neq 1$ of M fixed, then the group formed by adjoining H to M is Frobenius of type (h, m). Hence M is nilpotent if it satisfies (*) [cf. G. Higman, J. London Math. Soc. **32** (1957), 321–334; MR **19**, 633].

D. G. Higman (Ann Arbor, Mich.)

66:

Herstein, I. N. A remark on finite groups. Proc. Amer. Math. Soc. **9** (1958), 255–257.

Theorem: Let G be a finite group, A be an abelian subgroup of G. If A is a maximal subgroup of G, then G is solvable.

By mathematical induction the author reduces the theorem to the case that A is its own normalizer and has a trivial intersection with each conjugate distinct from itself. In this case, by a celebrated theorem of Frobenius, the set T, consisting of the identity element and the elements lying in no conjugate of A, is a normal subgroup of G. The inner automorphisms corresponding to elements of A induce on T an abelian group A^* of automorphisms, and each element of A^*, apart from the identity mapping, fixes only the identity element of T. From the relation of A^* to T (which the author formalizes in a lemma) he deduces first that T has, for every prime p dividing the order of T, a p-Sylow subgroup S_p which is mapped into itself by A^*. Indeed, the condition selects S_p uniquely for each p. Next he deduces that T is itself a p-group. But then T and $G/T \cong A$ are solvable, so G is solvable also.

R. H. Bruck (Madison, Wis.)

67:

Fröhlich, A. The near-ring generated by the inner automorphisms of a finite simple group. J. London Math. Soc. **33** (1958), 95–107.

If x and y are mappings of the additively written group G into itself, their product and sum may be defined by the equations $(x+y)g=xg+yg$ and $(xy)g=x(yg)$, for all $g \in G$. With these operations the set R composed of those mappings which are linear combinations with rational integral coefficients of the inner automorphisms of G is a distributively generated near-ring (DGN-ring). Here the author studies the DGN-ring R associated with a finite simple non-cyclic group G and proves that R coincides with $\bar{R}$, the set of all mappings of G into itself which leave the identity element fixed. Furthermore he shows that R', the multiplicative semigroup composed of all the distributive elements of R, is identical with the semigroup of endomorphisms of G. In proving these the author develops a structure theory for R analogous to the Artin-Wedderburn structure theory for a simple ring. He also considers the infinite case briefly, pointing out that R is properly contained in $\bar{R}$ if G is an infinite non-commutative simple group. Then he defines a topology for R and $\bar{R}$ in which $\bar{R}$ is the completion of R.

W. E. Deskins (East Lansing, Mich.)

68:

Armstrong, Ruth. Finite groups in which any two subgroups of the same order are isomorphic. Proc. Cambridge Philos. Soc. **54** (1958), 18–27.

The author studies the finite soluble groups in which any two subgroups of the same order are isomorphic. The principal result proved is that such a group G has a sequence $1 \leqq N \leqq M \leqq G$ of normal subgroups with the following properties: (i) the factors G/M, M/N, N are nilpotent and of pairwise coprime orders; (ii) G/M is cyclic, a Sylow subgroup of M/N is either cyclic or the quaternion group, and a Sylow subgroup of N is either elementary abelian or non-abelian of order p^3 and exponent p (p an odd prime).

In particular, the nilpotent length of G is $\leqq 3$. It is also proved that the derived length of G is $\leqq 4$, with strict inequality unless G contains a quaternion subgroup. These bounds are shown to be precise.

The proof of the main result depends on a detailed preliminary study of the groups of orders p^a and $p^a q^b$. The results for these special cases are carried over to the general case in a neat way by means of the Sylow systems of P. Hall [Proc. London Math. Soc. (2) **43** (1937), 316–323].

G. E. Wall (Sydney)

69:

Roman, Tiberiu. Sur les groupes de symétrie des ornements de bordure en relief. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957) no. 16, 9–14. (Romanian. French and Russian summaries)

The seven one-dimensional space groups, or ways to repeat a pattern on a ribbon [see, e.g., Coxeter, Trans. Royal. Soc. Canada (3) Sec. III **51** (1957), 2–13] were developed into a family of 31 groups by Speiser [Theorie der Gruppen von endlicher Ordnung, Springer, Berlin, 1927, p. 86]. For each crossing of lines in the pattern, Speiser distinguishes between crossing over and crossing under, as in a braid. The author sketches the history of these "$1\frac{1}{2}$-dimensional space groups" and gives a new notation for them.

H. S. M. Coxeter (Toronto, Ont.)

70:

Dieudonné, Jean. On simple groups of type B_n. Amer. J. Math. **79** (1957), 922–923.

Let E be a $(2n+1)$-dimensional vector space over a field K of characteristic 2, and $(e_i)_{-n \leq i \leq n}$ a basis of E over K. Let E_{ij} be the matrix such that $E_{ij}e_j=e_i$ and $E_{ij}e_k=0$ for $k \neq j$. The author proves that the $(2n+1) \times (2n+1)$ matrices

$$V_{i,t}=I+t^2E_{-i,i}, \quad W_{i,j,t}=I+(tE_{i,-j}-E_{j,-i}),$$

where $t \in K$, and i, j take the values $\pm 1, \pm 2, \cdots, \pm n$, generate the commutator subgroup of the orthogonal group defined by the quadratic form $Q(x)=\sum_{i=0}^{n} \xi_i \xi_{-i}$. This completes the identification started by the reviewer [Trans. Amer. Math. Soc. **84** (1957), 392–400; MR **19**, 247] of the simple groups defined by Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457]. The proof makes use of the general theory of quadratic forms established by the author.

Rimhak Ree (Vancouver, B.C.)

71:

Tamagawa, Tsuneo. On the structure of orthogonal groups. Amer. J. Math. **80** (1958), 191–197.

Très élégante et intéressante démonstration de la simplicité du groupe des commutateurs $P\Omega(V)$ d'un groupe orthogonal projectif $PO(V)$ relatif à une forme Q d'indice $\nu \geqq 1$, les cas exceptionnels habituels étant bien entendu exclus. Pour la première fois, il s'agit d'une démonstration valable aussi bien en caractéristique 2 qu'en caractéristique $\neq 2$; en outre, comme la démonstration d'Eichler [Quadratische Formen und orthogonale Gruppen,

Springer, Berlin, 1952; MR 14, 540] l'auteur démontre d'un seul coup le théorème pour toutes les dimensions, sans traiter d'abord le cas des dimensions 3 ou 4. Comme Eichler, l'auteur prend comme générateurs de $P\Omega(V)$ les produits $\rho_{x,u}$ de deux symétries correspondant à deux vecteurs non singuliers dans un même plan contenant un seul vecteur singulier x; mais alors qu'Eichler procède par de longs calculs matriciels, la démonstration de Tamagawa est entièrement géométrique. Son idée (inspirée d'une idée analogue d'Iwasawa dans l'étude du groupe linéaire [Proc. Imp. Acad. Tokyo **17** (1941), 57–59; MR **2**, 308]) est de considérer le groupe $P\Omega(V)$ comme opérant dans la quadrique $C: Q(x)=0$ de l'espace projectif. Il montre que $P\Omega(V)$ est un groupe primitif de transformations de C, d'où suit que tout sous-groupe distingué N de $P\Omega(V)$ est transitif sur C s'il n'est pas réduit à 1; on en conclut que si H_x est le sous-groupe (abélien) de $P\Omega(V)$ engendré par les $\rho_{x,u}$ correspondant à un même x, on a $NH_x=P\Omega(V)$, et par suite $P\Omega(V)/N$ est abélien. Mais comme on voit aisément que $P\Omega(V)$ est égal à son groupe des commutateurs, on a $P\Omega(V)=N$. *J. Dieudonné* (Evanston, Ill.)

72:

Ramanathan, K. G. On orthogonal groups. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1957**, 113–121.

The author gives a criterion (described below) for two classical groups of like type and dimension to be conjugate within the full linear group. As an application, he obtains generalizations of some 'local-global' theorems of T. Ono on the conjugacy of orthogonal groups over finite algebraic number fields [Nagoya Math. J. **9** (1955), 129–146; MR **17**, 582].

The author states his criterion in terms of simple algebras. Let F be a field of characteristic $\neq 2$, A a central simple algebra of finite rank over F. Then A is a total matrix algebra of, say, m-rowed matrices over a division algebra D. Let A have an involution $x\to x^*$ and suppose that, if the involution leaves every element of F fixed, then D is either F or a quaternion algebra over F. Denote by $G=G_m(D)$ the group of the non-singular elements of A. If $\alpha\in G$, denote by $O(\alpha)$ the subgroup of all $v\in G$ such that $v^*\alpha v=\alpha$. Call elements α, β of G multiplicatively equivalent if $v^*\alpha v=t\beta$ for some $v\in G$ and $t\in F$. The author's criterion is that if α, $\beta\in G$ and $\alpha^*=\varepsilon\alpha$, $\beta^*=\varepsilon\beta$ ($\varepsilon\neq 1$), then $O(\alpha)$, $O(\beta)$ are conjugate subgroups of G if, and only if, α, β are multiplicatively equivalent. The proof is quite short, the general case being deduced from the cases $m=1$, 2. A similar criterion is also proved for a given $O(\alpha)\subseteq G_m(D)$ to be 'conjugate' (in a natural sense) to a subgroup of a given $O(\beta)\subseteq G_n(D)$ ($m<n$).

G. E. Wall (Sydney)

73:

Straus, E. G.; and Szekeres, G. On a problem of D. R. Hughes. Proc. Amer. Math. Soc. **9** (1958), 157–158.

Let G be a group and p a prime. Define $H_p(G)$ to be the subgroup of G generated by the elements of G which do not have order p. The reviewer has conjectured the following: if $H_p(G)\neq 1$ and $H_p(G)\neq G$, then $[G:H_p(G)]=p$. The conjecture is easily demonstrated for $p=2$. Here it is shown that the conjecture is true for $p=3$. Besides utilizing the truth of Burnside's theorem for $p=3$, the proof depends upon the fact that 3 is sufficiently small to force an element of $H_p(G)$ to commute with any of its conjugates by elements not in $H_p(G)$.

D. R. Hughes (Chicago, Ill.)

74:

Barletta, Carmelo Longo. Contributo alla classificazione delle quasi schiere. Rend. Mat. e Appl. (5) **16** (1957), 506–510.

A system of elements u, a, b, c, $\cdots$ closed under a ternary operation (abc) which satisfies (A_1) $((abc)de)=(ab(cde))$, (B_1) $(auu)=a$, (B_2) $(aau)=u$, is called an abstract coset. If (A_1) is weakened to (A) $((auc)de)=(au(cde))$, the system may be called a quasi coset (quasi-schiera). In both cases the system forms a group with u as identity element under the group multiplication law $a\times b=aub$. With (A_1) it is known [J. Certaine, Bull. Amer. Math. Soc. **49** (1943), 869–877; MR **5**, 227] that $(abc)=a\times b^{-1}\times c$ but with (A) there may be other possibilities. It is shewn in the present paper that for a quasi coset with p elements there are at most $p^{(p-1)^2}$ ways in which closure of the ternary operations can be effected.

D. E. Rutherford (St. Andrews)

75:

Greco, Donato. Sui gruppi che sono somma di quattro o cinque sottogruppi. Rend. Accad. Sci. Fis. Mat. Napoli (4) **23** (1956), 49–59 (1957).

An extension to infinite groups and a revision for the finite case of the author's earlier result [Rend. Sem. Mat. Univ. Padova **22** (1953), 313–333; MR **15**, 503] is given.

D. G. Higman (Ann Arbor, Mich.)

76:

***Bruck, Richard Hubert. A survey of binary systems.** Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 20. Reihe: Gruppentheorie. Springer Verlag, Berlin-Göttingen-Heidelberg, 1958. viii+185 pp. DM 36.00.

The reader should be warned at the outset that the book is not as inclusive as the title might indicate. Hence a few general remarks concerning the scope of this book seem appropriate. Quite reasonably, the author chooses to exclude group theory and lattice theory from consideration. With these exceptions, the foreword announces "the area surveyed is the algebraic theory of systems with a single binary operation $\cdots$ The more highly developed parts of the subject have been treated in detail." Actually, the last two thirds of the text is devoted to loops and the initial part is a survey of other algebraic systems. An excellent bibliography (450 items) is included that supplements ideas that are not discussed in detail.

Chapter I begins with a single-valued mapping α from a subset $R(\alpha)$ of $G\times G$ into G (a binary operation defined on G). Now follow successive restrictions leading up to the associative law. This potentially dull dictionary of definitions for halfgroupoids, subgroupoids, groupoids, quasigroups, loops, is accompanied and enlivened by a discussion of the concepts of "freeness" for these systems. The chapter concludes with a discussion of certain relations between binary systems and geometry (specifically, nets and projective planes) and a brief survey of more general n-ary systems.

Chapter II introduces the associative law. The treatment of semigroups contains a well-organized and detailed account of the ideal structure of semigroups. Otherwise, the reader must be content with little more than the bibliography. Groups enter only to the extent of their relation to other binary systems with a brief account of groupoids with a single identcial relation defining subclasses of groups. Homomorphic imbedding, Brandt groupoids, mixed groups, multigroups, hypergroups all receive brief attention. A set G with a single n-ary

operation defined on G (polyadic groups) is considered with a discussion of Post's Coset Theorem.

Beginning with Chapter III and the isotopy of binary systems, the book is almost a text on loop theory rather than a survey. It would appear that this was the author's original intention and that the survey of other algebraic systems was an afterthought. Many of the sections on loop theory represent either new material or applications of new methods to obtain improvement of older treatments. Particular examples of this are the sections on normal endomorphisms and their application to the direct decomposition of operator loops; the improvement of the author's earlier treatment of nilpotency of loops and the discussion of local properties.

Chapters VII and VIII (roughly one third of the text) are devoted to the theory of Moufang loops. Here the survey becomes particularly detailed and new results on an invariant of the class of all isotopic Moufang loops, the core of a Moufang loop, add to the structure theory. Moreover, the last chapter on commutative Moufang loops has not previously appeared in print. It is shown that there exist commutative Moufang loops that are centrally nilpotent of any class k. The author then proves: "Let n be a positive integer, $n \geqq 3$. Then every commutative Moufang loop G which can be generated by n elements is centrally nilpotent of class at most $n-1$."

The author has been careful throughout the survey of loops to ignore those aspects that are straightforward analogues of group theory.

The reviewer feels that the principal criticism to be raised about the book is with the last chapter. It seems questionable, in view of the complicated nature of the proofs and the results, to think of the theory of commutative Moufang loops as being "highly developed". In addition to the ideal structure of semigroups, other areas of semigroups that have been relegated to the bibliography are more deserving of this description.

Overall, this work will be of considerable value to mathematicians since it brings together (with an excellent bibliography) much otherwise widely scattered material on binary systems.

L. J. Paige (Los Angeles, Calif.)

77:

Sade, A. Quelques remarques sur l'isomorphisme et l'automorphisme des quasigroupes. Abh. Math. Sem. Univ. Hamburg **22** (1958), 84–91.

Part I of the paper is best summarized by the title of the paper. In Part II the author seems to be supposing that he has before him two finite quasigroups and wishes to know whether they are isomorphic. He gives a variety of easily applied necessary conditions, involving various types of single-valued mappings of a quasigroup into itself. The most simply described of these mappings is the one which carries each element into its square.

R. H. Bruck (Madison, Wis.)

78:

Sade, Albert. Quasigroupes automorphes par le groupe linéaire et géométrie finie. J. Reine Angew. Math. **199** (1958), 100–120.

Ce travail est le 3ième papier publié par l'auteur sur les quasigroupes dont le groupe d'automorphisme admet un diviseur donné [cf. Canad. J. Math. **9** (1957), 321–335; **10** (1958), 294–320; MR **19**, 389]. Il contient le développement des idées formulées pour la première fois dans "Quasigroups" [Marseille, 1950; MR **13**, 203], puis dans les C. R. Acad. Sci. Paris **237** (1953), 420–422 [MR **15**, 98]. Un quasigroupe est un ensemble muni d'une loi de composition binaire, partout définie, homogène et satisfaisant à la loi du quotient bilatère ($ax=b$ et $ya=b$ ont une solution et une seule en x et y). L'ensemble des substitutions linéaires $(x \to xm+h) \pmod n$, $(m, n)=1$, est un groupe L_n que l'on peut appeler le groupe linéaire de Z/n. Un quasigroupe $Q(\times)$, défini sur l'anneau Z/n, est automorphe par le groupe linéaire de cet anneau, L_n, si, pour tous $x, y, z, h \in Z/n$ et m premier avec n, on a $x \times y = z \Leftrightarrow (mx+h) \times (my+h) = mz+h \pmod n$. La première partie est consacrée aux propriétés de structure de tels quasigroupes (parastrophie, loi de composition, nombre de solutions, diviseurs, quasigroupe quotient...). Le principal théorème est: Tout quasigroupe $Q(\times)$, d'ordre n impair, automorphe par L_n, a pour loi de composition $x \times y = (x-y)g(d)+y \pmod n$, où $d=(x-y, n)$ et où g est une fonction arbitraire de d, définie sur l'ensemble des diviseurs de n, prenant ses valeurs dans Z/n et telle que $(g, n)=(g-1, n)=1$. Le dénombrement des quasigroupes d'ordre n, automorphes par le groupe linéaire, résulte d'un lemme qui généralise la notion d'indicateur, et peut aussi être obtenue par une formule de LeVêque-Touchard. On trouve pour le nombre de tels quasigroupes $N=\prod P(n/d)$, où d parcourt tous les diviseurs de n, y compris l'unité, et où $P(n)=n(p-2)(q-2)(r-2)\cdots/(pqr\cdots)$ est le nombre des entiers naturels u, non supérieurs à $n=p^A q^B r^C \cdots$ et tels que u et $u-1$ soient premiers avec n, où $p, q, r \cdots$ sont les facteurs premiers inégaux de n. Les quasigroupes Q automorphes par le groupe linéaire jouissent encore de cette propriété, que les d complexes $R_i=(i, d+i, 2d+i, \cdots, n-d+i) \pmod n$, $i=0, 1, 2, \cdots, d-1$, sont des diviseurs disjoints de Q; R_0 est automorphe par $(x \to mx)$, $(m, n)=1$, et l'ensemble $G=(0, 1, 2, \cdots, n/d-1)$, muni de la loi $x \bullet y = z \Leftrightarrow dx \times dy = dz$, est un quasigroupe automorphe par $(x \to x+h) \pmod{n/d}$ et $(x \to mx)$, $(m, n)=1$. On déduit de là la condition de normalité de R_0 et un système de représentants pour le quasigroupe quotient. Des exemples illustrent la théorie.

La seconde partie contient l'exposé d'une interprétation géométrique des quasigroupes $Q(\times)$, tout point étant l'ensemble des trois coordonnées, x, y, z telles que $z=x \times y$. Cette étude, qui pourrait conduire à des conclusions plus générales, est limitée ici à $Q(\times)=QL$, où la loi de composition est de la forme $x \times y = ax+(1-a)y$, sur Z/n. L'alignement est défini comme en géométrie usuelle. La géométrie contient n^2 points, dont l'ensemble est noté GQL. Toute droite contient exactement n points et deux points quelconques, distincts, d'une droite, (x, y, z) et (x', y', z'), satisfaisant à la condition $(x-x', y-y', z-z', n)=1$, définissent cette droite. Par tout point passent $\Delta = n(p+1)(q+1)(r+1)\ldots/(pqr\cdots)$ droites distinctes et le nombre total des droites est $n\Delta$. Dans le cas général deux points définissent seulement une sous-droite et le nombre des droites ayant en commun cette sous-droite est un diviseur de Δ. Le nombre des droites issues d'un point donné et coupant une droite donnée (i) en d points, (ii) en un nombre quelconque de points, (iii) en 0 points, est explicité. Tout automorphisme de QL transforme une droite D en une droite parallèle à D, le parallélisme étant défini comme en géométrie usuelle. Tout endomorphisme de QL projette D sur une sous-droite portée par une parallèle à D; on détermine dans ce cas les endomorphismes qui projettent D dans lui-même; en particulier il existe une droite qui reste invariante par les $n\varphi(n)$ automorphismes de QL. Pour que l'ensemble $Q_g(\cdot)$ des points d'un quasigroupe $Q(\times)$, muni de la loi de composition $(x, y, z) \cdot (x', y', z') = (x \times x', y \times y', z \times z')$, soit un quasigroupe il faut et il suffit que $Q(\times)$ soit entropique

$(ab\cdot cd=ac\cdot bd)$; alors deux points et leur produit sont en ligne droite et les n points d'une droite quelconque forment un diviseur de Q_g isomorphe à QL; toute droite D est diviseur normal de Q_g, les cosets sont les parallèles à cette droite; le système des représentants est formé des n points d'intersection des n parallèles avec une sécante coupant la première droite en un seul point; il est isomorphe à QL et au quasigroupe quotient Q_g/D. On aboutit à des théorèmes analogues en considérant le faisceau des droites issues d'un point donné. Le papier se termine par l'étude des isotopies et autotopies de QL. Dans le cas où n est premier, QL définit un balanced incomplete block design de Yates. *P. Vincensini* (Marseille)

79:

Osborn, J. Marshall. A theorem on A-loops. Proc. Amer. Math. Soc. **9** (1958), 347–349.

The author proves that every commutative diassociative A-loop is also a commutative Moufang loop. This settles a conjecture of R. H. Bruck and the reviewer. *L. J. Paige* (Los Angeles, Calif.)

80:

Belousov, V. D. Derivative operations and associators in loops. Mat. Sb. N.S. **45(87)** (1958), 51–70. (Russian)

The author defines a loop A to be of type (L, C, R) if the left, middle and right associator subgroups of A are respectively L, C and R. It is then shown that there exists a loop A of type (L, C, R), where L, C, and R are arbitrarily assigned groups. Next, despite the notation, the well-known result that isotopic loops are of the same type is obtained. The converse for loops defined on the same set of elements is also derived. Finally, elementary results are obtained for those loops which have the property that all principal isotopes are isomorphic. *L. J. Paige* (Los Angeles, Calif.)

TOPOLOGICAL GROUPS AND LIE THEORY

81:

Borel, Armand. Transformation groups with two classes of orbits. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 983–985.

The author announces several results, whose proofs, outlined briefly in the paper, will appear in detail later. Let G be a compact connected Lie group acting effectively on a locally compact space X. The orbits of two points x and y are said to be equivalent if G_x and G_y (their isotropy groups) are conjugate. Let X_u denote the subspace of X of points whose orbits are equivalent relative to the subgroup U, and F_u the set of fixed points of U. Two locally trivial fiberings of X_u are obtained with structural group N_u/U. Besides the obvious, there is a fibering with G/N_u as base and $F_{u'}$, U' conjugate to U, as fibres. This and another more technical proposition, are used in establishing the following results: (1) If X is an n-sphere, and G has only one class of orbits on X, then, if G is not transitive, it is isomorphic to U(1) or Sp(1), the isotropy groups are reduced to the identity, and $n+1$ is divisible by 2 or 4. (2) If X has the local and global integral cohomology properties of a sphere and is a generalized orientable manifold over the integers mod p for every prime p and G has exactly two classes of orbits over X, then one consists of fixed points. (3) Let X be a connected n-dimensional differentiable manifold on which G operates differentiably. If the subspace F of fixed points is non-empty and G has one class of orbits in $X-F$, then these orbits are homeomorphic to S_m. If $m=n-\dim F-1$, then $G=\mathrm{U}(1)$ and the isotropy groups in $X-F$ are reduced to the identity. If X has the integral cohomology of Euclidean n-space, so has F. *P. S. Mostert* (New Orleans, La.)

82:

Mautner, F. I. Spherical functions over $\mathfrak{P}$-adic fields. I. Amer. J. Math. **80** (1958), 441–457.

Let Ω be a field, complete under a discrete valuation such that the residue field $\mathfrak{O}/(\pi)$ is a finite field with q elements ($\mathfrak{O}$=the valuation ring, (π)=the prime ideal of $\mathfrak{O}$ with generator π). As usual, let $G=\mathrm{PGL}(2, \Omega)$ denote the factor group of $\mathrm{GL}(2, \Omega)$ modulo its center and let K and A be the canonical images in G of the subgroup $\mathrm{GL}(2, \mathfrak{O})$ of $\mathrm{GL}(2, \Omega)$ and of the subgroup of all diagonal matrices in $\mathrm{GL}(2, \Omega)$, respectively. A complex-valued function f on G is called a spherical function if f is constant on every double coset of G mod K. Following the analogy of classical (complex) semi-simple Lie groups, the author studies the properties of such spherical functions on G.

Let S be the space of all spherical functions on G, S^0 the subspace of all spherical functions of compact support and S^p ($p\geqq 1$) the intersection of S with the L^p-space on G relative to Haar measure. For any complex number s, let α_s denote the character of A such that $\alpha_s(A\cap K)=1$, $\alpha_s(y)=q^{s-1}$, where y is the element of G determined by the diagonal matrix with diagonal elements $1, \pi$. Then each α_s defines a representation, $g\to T_s(g)$, of G on a Hilbert space which corresponds to representations of the principal series in the classical case. For f in S^0, let $F(s)$ be the Fourier transform of f, namely, the trace of the operator $\int_G f(g)T_s(g)dg$. Then the author proves that the mapping $f\to F$ gives an isomorphism of the algebra S^0 onto the algebra of all even Fourier polynomials of the form $\sum c_n(q^{n(s-\frac{1}{2})}+q^{-n(s-\frac{1}{2})})$ and that, in general, the Fourier transform $F(s)$ of any f in S^1 is a regular analytic function of s on $0<\mathrm{R}(s)<1$ satisfying the functional equation $F(s)=F(1-s)$. Furthermore, all maximal ideals of S^1 are determined and the Plancherel formula for S^2 is also established. *K. Iwasawa* (Cambridge, Mass.)

83:

Bredon, Glen E. Some theorems on transformation groups. Ann. of Math. (2) **67** (1958), 104–118.

The paper is divided into two parts. In the first part the author studies the local structure of an orbit of a locally compact finite-dimensional group acting on a Hausdorff space. The main theorem: A locally compact group G which acts effectively on a connected manifold M is a Lie group if and only if for every point x of M and every neighborhood U of the identity of G there is a neighborhood $V\subset U$ of the identity such that $V(x)$ is locally connected. In the second part the author studies locally compact finite-dimensional transformation groups of manifolds. The main theorem: Any compact group which acts effectively on an n-manifold is a Lie group if there is an orbit of dimension $n-1$ or $n-2$. *C. T. Yang* (Philadelphia, Pa.)

84:

Conner, P. E. A note on a theorem of Mostow. Proc. Amer. Math. Soc. **9** (1958), 467–471.

Let π be a group acting properly (hence without fixed points) on a manifold M^n. By a simple argument with spectral sequences the author shows that if π is abelian of rank m and $H^i(M^n; Z)=0$ for $i>n-m$, then one of the

following holds: $H^{n-m}(M^n; Z)$ is not finitely generated; $H^{n-m}(M^n; Z)=0$ and M^n/π is open; $H^{n-m}(M^n; Z)\cong Z$ or Z_2 and M^n/π is closed. A similar theorem for finitely generated solvable groups is used to prove the following: Let H be a closed subgroup of a simply connected Lie group G. If $\pi_1(G/H)$ is solvable then rank $\pi_1(G/H)\leq s-t$ and equality holds if and only if G/H is compact. Here s and t are defined by the known decompositions $G=L\times E^s$, H (identity component)$=L'\times E^t$, where L and L' are maximal compact subgroups.

P. A. Smith (New York, N.Y.)

FUNCTIONS OF REAL VARIABLES

See also 93, 275.

85:

Nikol'skiĭ, S. M. Imbedding theorems for functions with partial derivatives, considered in different metrics. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 35–37. (Russian)

This note formulates some existence theorems for functions f of several real variables; the conditions involve partial L^p norms of some of the partial derivatives of f. No abstract of this abstract can be appreciably shorter than the note itself and give any clear idea of its contents.

M. M. Day (Urbana, Ill.)

86:

Gál, István S. On the fundamental theorems of the calculus. Trans. Amer. Math. Soc. **86** (1957), 309–320.

The fundamental theorem of the differential calculus (asserting that if f is continuous in the closed interval $[a, b]$ and if $f'=0$ in the interior of $[a, b]$ then f is a constant) has been generalized by several writers. The following theorem, proved in the present paper, includes these earlier generalizations as special cases. Theorem: Let f be a real-valued function on a finite closed real interval $[a, b]$ such that (1) for every x in $[a, b]$, $\limsup_{\xi\to x-} f(\xi)\leq f(x)\leq \limsup_{\xi\to x+} f(\xi)$, (2) the upper right derivative D^+f is non-negative almost everywhere in $[a, b]$, (3) $D^+f>-\infty$ in $[a, b]$ except possibly on a countable set. Then $f(a)\leq f(b)$.

The proof is an elementary (though not particularly short) one based on explicit use of the compactness of the finite closed interval. Where, for $-\infty\leq t\leq 0$, $\mu(t)$ denotes the exterior measure of the set of all x in $[a, b]$ for which $D^+f(x)<t$, this method is also used to extend the above theorem by replacing hypothesis (2) by the requirement $\mu(t)\to 0$ as $t\to-\infty$, and simultaneously replacing the conclusion by

$$f(b)-f(a)\geq \int_{-\infty}^{0} t\,d\mu(t).$$

A few additional results and applications to Perron integration are noted.

T. A. Botts (Charlottesville, Va.)

87:

★Mitrinović, D. S. Važnije nejednakosti. [Important inequalities.] Matematička Biblioteka, No. 7. Univerzitet u Beogradu, Belgrade, 1958. 64 pp.

Fifteen pages of various introductions are followed by two main parts: a twenty-two page collection of inequalities with proofs (absolute values, various means, Abel's convergence test and convex functions), and a thirty-five page collection of inequalities without proofs, culled from a dozen texts, such as Crosland's "Upper school algebra", Hardy's "A course in pure mathematics", and some Russian collections. In spite of the booklet's title, Schwarz' (Bunyakovsky's) inequality appears in one line, Hölder's and Minkowski's not at all.

Z. A. Melzak (Montreal, Q.)

88:

Kober, H. On the arithmetic and geometric means and on Hölder's inequality. Proc. Amer. Math. Soc. **9** (1958), 452–459.

If $n\geq 2$, assume $0<q_1\leq q_2\leq\cdots\leq q_n$, $\sum q_i=1$, $x_i\geq 0$; let $\Delta_n=\sum_1^n q_i x_i-x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}$, $S_n=\sum_{1\leq i<k\leq n}(x_i^{\frac{1}{2}}-x_k^{\frac{1}{2}})^2$. The author derives a two-sided comparability theorem for Δ_n and S_n in the form of the inequality

$$(*)\qquad q_1(n-1)^{-1}\leq \Delta_n/S_n\leq q_n.$$

Corresponding problems for Hölder's and Minkowski's inequalities are also discussed.

The lower bound in (*) is best possible and in special cases the upper bound q_n cannot be decreased. However, it seems to be an open question whether q_n is the least upper bound in general. When the point $(x_1, \cdots, x_n)$ in n-space $(n>2)$ tends to $(x_0, x_0, \cdots, x_0)$, where $x_0>0$, the fraction Δ_n/S_n tends to a unique limit if, and only if, $q_1=q_2=\cdots=q_n=1/n$. This limit is $2/n^2$.

T. M. Apostol (Pasadena, Calif.)

89:

Myrberg, Pekka J. Sur une généralisation de la moyenne arithmétique-géométrique de Gauss. C. R. Acad. Sci. Paris **246** (1958), 3201–3204.

90:

Tukey, John W. A smooth invertibility theorem. Ann. Math. Statist. **29** (1958), 581–584.

Good behavior of a Jacobian does not alone guarantee invertability of a mapping in the large. But when the domain of the mapping is open and connected (the author redundantly says "arcwise connected"), when the range is simply connected, and when the inverse image of compact sets is compact, then all is well.

L. J. Savage (Rome)

91:

Nikitin, I. I. Functions which are convex with respect to a given family of curves. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 49–82. (Russian)

92:

Wintner, Aurel. On cosine-like arches. Amer. J. Math. **80** (1958), 125–130.

A cosine-like arch is a completely convex function on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The author finds the following necessary condition for a cosine-like arch $f(x)$: $f(x)=q(\sin x)$, where $q(w)=(1-w^2)^{\frac{1}{2}}p(w)$, $q(0)\geq 0$, $p(w)=\sum_{n=0}^{\infty}\mu_n w^n$, $\mu_n=\int_{-1}^{1} t^n d\phi(t)$, ϕ nondecreasing, and $q(\sin z)$ is an entire function of exponential type. This follows from the following fact: let $f(z)$ be regular in the strip $|x|<\frac{1}{2}\pi$ and real-valued on the real axis; then f has a nonnegative real part if and only if there is a nondecreasing ϕ such that, with $p(w)=\int_{-1}^{1}(1-zw)^{-1}d\phi(t)$ in the plane with the rays $(-\infty, -1]$ and $[1, \infty)$ deleted, we have $f(z)=p(\sin z)\cos z$. This in turn is a consequence of the Carathéodory-Toeplitz criterion for functions with positive real part.

R. P. Boas, Jr. (Evanston, Ill.)

MEASURE AND INTEGRATION

93:

Marcus, Solomon. Fonctions convexes et fonctions internes. Bull. Sci. Math. (2) **81** (1957), 66–70.

A set E of real numbers is said to be a median set (ensemble médian) if $\frac{1}{2}(x+y)$ belongs to E whenever x and y belong to E. It is proved that every median and frontier set has Lebesgue interior measure zero. This lemma forms a common foundation for generalizations of theorems of Ostrowski and Császár on the non-measurability of discontinuous convex functions and of non-monotonic internal functions (fonctions internes) [A. Császár, Fund. Math. **36** (1949) 72–76; MR **11**, 89].

F. F. Bonsall (Newcastle-upon-Tyne)

94:

Berg, Lothar. Massbestimmung linearer Punktmengen. Math. Nachr. **16** (1957), 195–205.

In his thesis [Math. Nachr. **14** (1955), 263–285; MR **18** 645] the author studied an extensive class of expansions for real numbers, which includes the well-known expansions of Lüroth, Sylvester and Engel. In particular he asked himself whether in certain special cases an analogue of F. Bernstein's theorem for the continued fraction algorithm holds (any positive integer k appears an infinity of times in the sequence of quotients in the continued fraction algorithm of almost any real number).

In this paper the author investigates three special cases of the general theory and for these he solves the above problem. One of his results reads: "Let $x=1/a_1+1/a_2+1/a_3+\cdots+1/a_n+\cdots$ denote the Sylvester expansion of x $(0\leqq x\leqq 1)$, so that $a_n\geqq a_n^*$, where $a_n^*=a_{n-1}(a_{n-1}-1)+1$. Let M_k for any integer $k\geqq 0$ denote the set of numbers x such that $a_n=a_n^*+k$ holds at most for a finite number of values for n. Then M_k has measure 1 for every k." Hence in this case the analogue of Bernstein's theorem does not hold.

J. Popken (Amsterdam)

95:

Bauer, Heinz. Sur l'existence de mesures avec une image donnée. C. R. Acad. Sci. Paris **246** (1958), 1953–1955.

Etant donné deux espaces localement compacts X, Y et une application continue φ de X dans Y, on donnera des conditions suffisantes pour qu'une ou toute mesure positive sur Y soit l'image de mesures positives sur X par l'application φ.

Résumé de l'auteur

96:

Besicovitch, A. S. A uniqueness theorem and a problem on integration. J. London Math. Soc. **33** (1958), 82–84.

Let $f(x)$ be a real-valued function on the real interval $(0, \infty)$, and let $\varphi(x, t)$ be a non-zero real-valued function defined on the set of all pairs (x, t) of reals for which $x>t\geqq 0$. It is proved that if (i) $f(x)$ is Lebesgue integrable over each interval $(0, a)$ for which $0<a\leqq\infty$, and (ii) for each $t\geqq 0$, $\varphi(x, t)$ is a monotone absolutely continuous function of x on each interval (t', h) for which $t<t'<h<\infty$, and $\lim_{x\to\infty}\varphi(x, t)=0$, and (iii) for each $t\geqq 0$, $f(x)\varphi(x, t)$ is a Lebesgue integrable function of x on each interval (t, t') for which $t<t'\leqq\infty$, and for almost all $t>0$ $\int_t^\infty f(x)\varphi(x, t)dx=0$, then $f(x)=0$ for almost all $x>0$.

As an application, it is shown, generalizing a problem posed by D. J. Newman [Amer. Math. Monthly **64** (1957), 49], that if $f(x, y)$ is a Lebesgue measurable function on the real plane such that $\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}|f(x, y)|dy]dx<\infty$ and such that, for every line l, $\int_l f(x, y)ds=0$, then $f(x, y)=0$ almost everywhere.

{It has been pointed out to the reviewer that the last-quoted result was known earlier, with the following simple proof: For each $(\xi, \eta)\neq(0, 0)$ in the plane, the Fourier transform

$$\hat{f}(\xi, \eta)=\int\int e^{i(\xi x+\eta y)}f(x, y)dxdy$$

of the L_1 function f may, by the rotation of axes $\xi x+\eta y=\zeta u$, $\eta x-\xi y=\zeta v$ $(\zeta^2=\xi^2+\eta^2)$, be put in the form $\int e^{i\zeta u}\int f^t(u, v)dv$, in which $f^t(u, v)$ is $f(x, y)$. Then $\int f^t(u, v)dv$ is $\int_l f(x, y)ds$ over a certain line l, hence is zero. It follows that the continuous function $\hat{f}$ vanishes identically, whence f is zero almost everywhere.

The reviewer does not know the origin of the theorem. For a related result, see H. Cramér and H. Wold, J. London Math. Soc. **11** (1936), 290–294.}

T. A. Botts (Charlottesville, Va.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 31, 92, 170, 171, 194, 197, 226.

97:

Onicescu, O. Quelques remarques sur la solution du système de Cauchy pour l'intérieur d'une courbe de Jordan, comme un problème de Cauchy. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 10, 9–13. (Romanian. Russian and French summaries)

The Cauchy-Riemann equations for u, v are replaced by an equivalent vector differential equation, involving a matrix differential operator. A solution with prescribed values on a circle, or on a general convex closed curve, is obtained by iteration.

W. Kaplan (Ann Arbor, Mich.)

98:

Tumarkin, G. C.; and Havinson, S. Ya. On the removing of singularities for analytic functions of a certain class (class D). Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 193–199. (Russian)

A natural extension of the class of analytic functions bounded in a domain D is the class $H_p(D)$. A single-valued analytic function $f(z)$ belongs to $H_p(D)$ $(0<p<\infty)$ if the subharmonic function $|f(z)|^p$ has in D a harmonic majorant. For such functions Parreau [Ann. Inst. Fourier Grenoble **3** (1951), 103–197; MR **14**, 263] and Rudin [Trans. Amer. Math. Soc. **78** (1955), 46–66; MR **16**, 810] proved: (*) If N is a compact set of capacity 0 in the domain D and $f(z)\in H_p(D-N)$, then $f(z)$ can be defined on N so that $f(z)$ will be analytic in the whole of D and $f\in H_p(D)$.

The class $H_p(D)$ $(0<p<\infty)$ is a subclass of the class $\mathrm{Log}^+ D$, to which $f(z)$ belongs if $\log^+|f(z)|$ has in D a harmonic majorant. The authors denote the domain by G and this class by $A(G)$. Theorem (*) is not true for $A(G)$.

The authors define a class $D(G)$, intermediate between all $H_p(G)$ and $A(G)$. A single-valued analytic function $f(z)$ in G is $\in D(G)$ if the functions $\log^+|f(z)/\mu|$ (subharmonic in G) have in G best harmonic majorants $u^\mu(z)$ such that $\lim_{\mu\to\infty}u^\mu(z_0)=0$, $z_0\in G$.

Two theorems give necessary and sufficient conditions for $f(z)$ to belong to $D(G)$. The following theorem is then proved. Let G be an arbitrary domain and N a compact subset of G with logarithmic capacity zero. If $f(z)\in$

$D(G-N)$, then $f(z)$ can be defined on N so that the (continued) function $f(z) \in D(G)$. The theorem of Parreau and Rudin is a simple corollary of this theorem.

B. A. Amirà (Jerusalem)

99:

Raljević, Š. Remarque sur un théorème de M. Marden. Srpska Akad. Nauka. Zb. Rad. **55**, Mat. Inst. **6** (1957), 69–72. (Serbo-Croatian. French summary)

This note shows by example that Theorem (44.1) is incorrect as stated in the reviewer's "The geometry of the zeros of a polynomial in a complex variable", Math. Surveys, no. 3, Amer. Math. Soc., N.Y., 1949 [MR **11**, 101], p. 157. As the reviewer has known for some time, the statement in the theorem "$f(z)$ has $n-k$ zeros on the unit circle" should read "$f(z)$ has $n-k$ zeros on or symmetric in the unit circle". The other zeros mentioned in the theorem are in addition to these zeros. See also F. F. Bonsall and M. Marden, Proc. Amer. Math. Soc. **3** (1952), 471–475; **5** (1954), 111–114 [MR **13**, 938; **15**, 613].

M. Marden (Milwaukee, Wis.)

100:

Gumennïĭ, P. V. A method for approximation of complex roots of an algebraic equation. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 98–102. (Ukrainian)

101:

Nishimiya, Han. On a coefficient problem for analytic functions typically-real in an annulus. Kōdai Math. Sem. Rep. **9** (1957), 59–66.

Let $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$ be regular and typically-real in an annulus $q<|z|<1$ and satisfy $\operatorname{Im} f(z)\cdot \operatorname{Im} z>0$ for $\operatorname{Im} z\neq 0$. Then, as shown by the reviewer [Kōdai Math. Sem. Rep. **9** (1957), 42–48; MR **19**, 404], the precise estimation

$$-\frac{n}{1-q^{2n}}\left(\frac{k_n}{n}(a_1-a_{-1})+(a_1q-a_{-1}q^{-1})q^n\right)\leqq a_n \leqq \frac{n}{1-q^{2n}}\left(a_1-a_{-1}+\frac{k_n}{n}(a_1q-a_{-1}q^{-1})q^n\right) \quad (n=\pm 1, \pm 2, \cdots)$$

holds good where k_n is defined by $k_{\pm 1}=\pm 1$, $k_n=-\min_{0\leq\varphi\leq\pi}(\sin n\varphi/\sin\varphi)$, $k_{-n}=-k_n$ $(n=2, 3, \cdots)$. The present author gives an alternative proof of this result by means of an integral representation of the coefficients and further determines the totality of extremal functions completely.

Y. Komatu (Tokyo)

102:

Piranian, G.; and Thron, W. J. Convergence properties of sequences of linear fractional transformations. Michigan Math. J. **4** (1957), 129–135.

The point set E in the extended complex plane is called a set of convergence if there exists a sequence $\{T_n\}_{n=1}^{\infty}$ of nonsingular linear fractional transformations such that (1) $\{T_n(z)\}_{n=1}^{\infty}$ converges if z belongs to E and diverges if z does not belong to E. If E is a set of convergence and (1) converges to $f(z)$ for z in E, then f is a nonsingular linear fractional transformation or f maps E onto a set containing at most two points. The set F is a set of divergence (SD) if its complement is a set of convergence. Several theorems concerning sets of divergence are proved, e.g., every SD is of type $G_{\delta\sigma}$; every set of type G_δ is a SD; every countable subset of a straight line is a SD; not every SD is of type G_δ; not every countable set is a SD; if the closure of the set E is countable, E is a SD.

H. S. Wall (Austin, Tex.)

103:

Edrei, Albert. Power series having partial sums with zeros in a half-plane. Proc. Amer. Math. Soc. **9** (1958), 320–324.

Let $1+a_1z+a_2z^2+\cdots$ be a formal power series which does not reduce to a polynomial. If the zeros of every partial sum $s_n(z)=1+a_1z+\cdots+a_nz^n$ lie in some angle $|\theta_n-\arg z|\leqq\frac{1}{2}\alpha$ of fixed aperture $\alpha<2\pi$ then the power series represents an entire function of order zero: $\lim |a_n|^{1/n \log n}=0$. This is a special case of results of Rosenbloom [Lectures on functions of a complex variable, Univ. of Michigan Press, Ann Arbor, 1955, pp. 265–285; MR **17**, 246] and the reviewer [Duke Math. J. **18** (1951), 573–592; MR **13**, 222]; the result is implicit in a much older paper by Carlson [C. R. Acad. Sci. Paris **179** (1924), 1583–1585]. The present author shows that in the case $\alpha=\pi$ one has the much sharper estimate $\limsup |a_n|^{1/n^2}<1$. He gives an example which shows that the exponent $1/n^2$ cannot be replaced by a function which decreases more rapidly, even in the case $\alpha=0$.

J. Korevaar (Madison, Wis.)

104:

Dinghas, Alexander. Bemerkungen zum Phragmén-Lindelöfschen Prinzip. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 59–64.

Let $w(z)$ be holomorphic in the half-plane $\operatorname{Re} z>0$ and satisfy the condition, $\lim\sup_{z\to z_0}|w(z)|\leqq 1$, for every z_0 with $\operatorname{Re} z_0=0$. Set

$$\psi_n(\theta)=\begin{cases}\cos n\theta, & n \text{ odd},\\ \sin n\theta, & n \text{ even},\end{cases}$$

$$P(r^{-1},\theta)=\frac{a_1}{r}\psi_1(\theta)+\cdots+\frac{a_n}{r^n}\psi_n(\theta)\ (a_k \text{ real}),$$

$$\mu(r)=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\log^+|w(re^{i\theta})|P(r^{-1},\theta)d\theta, \quad z=re^{i\theta}.$$

The author gives two proofs for the theorem that if $P(r^{-1},\theta)>0$ for $r>0$ and $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$, either $|w(z)|\leqq 1$ in $\operatorname{Re} z>0$ or, from a certain value of r on, $\mu(r)>0$ and non-decreasing. As the author points out, the special case $P(r^{-1},\theta)=r^{-1}\cos\theta$ was first established by Ahlfors [Trans. Amer. Math. Soc. **41** (1937), 1–8]. It is shown, moreover, that if $P>0$ on an interval $[r,\rho]$ $(0<r<\rho)$, then $\mu(r)$ remains constant on $[r,\rho]$ if and only if the set $\{z\,|\,|w(z)|=1, r<|z|\leqq\rho\}$ is empty. Extensions of the above are considered when (1) $w(z)$ is holomorphic in any simply connected region G of the z-plane which possesses $z=0, \infty$ as boundary points and (2) $w(z)$ satisfies the condition $\lim\sup_{z\to\zeta}|w(z)|\leqq 1$, where $z\in G$, for every finite boundary point ζ of G. Other extensions are briefly indicated.

W. Seidel (Notre Dame, Ind.)

105:

Gaier, Dieter. Über ein Iterationsverfahren von Komatu zur konformen Abbildung von Ringgebieten. J. Math. Mech. **6** (1957), 865–883.

In his preparatory note with the same title [Z. Angew. Math. Mech. **36** (1956), 252–253; MR **18**, 882] the author has indicated a quantitative estimate for the rate of convergence of an iterative process introduced by the reviewer [Proc. Japan Acad. **21** (1945), 146–155; MR **11**, 341] which gives a conformal mapping of a doubly-connected ring domain onto an annulus. In the present note detailed proofs are given for the above-mentioned estimate as well as related results.

Y. Komatu (Tokyo)

106:

Myrberg, Lauri. Über meromorphe Funktionen und Kovarianten auf Reimannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I, no. 244 (1957), 18 pp.

f sei eine meromorphe Funktion auf einer Riemannschen Fläche F, F_n ein kompaktes Teilgebiet von F und z_0 ein Punkt in F_n. Der Autor definiert den Wert $T(f, F_n, z_0)$ der charakteristischen Funktion von f auf F_n durch

$$T(f, F_n, z_0)=(2\pi)^{-1}\int_{\partial F_n}\log^+|f(\zeta)|dh(\zeta, F_n, z_0)$$
$$+\sum_{x_k\in F_n} g(x_k, F_n, z_0)$$

wo ∂F_n=Rand von F_n, $g(x, F_n, z_0)$ die Green'sche Funktion von F_n mit dem Pol in z_0, h die konjugierte Funktion und x_k die Pole von f bedeuten. Auf einer Ausschöpfung $F_1\subset F_2\subset\cdots(z_0\in F_1)$ wächst $T(f, F_n, z_0)$ monoton, und durch die Bedingung

$$\limsup_{n\to\infty}\frac{T(f, F_n, z_0)}{\iint_{F_n} g(z, F_n, z_0)\Delta\log K(z, z)d\sigma_z}<\infty$$

wo K den Bergman'schen Kern darstellt, wird ein Körper $\mathfrak{F}$ von meromorphen Funktionen f und eine Klasse Φ von meromorphen Kovarianten φ auf der Fläche definiert, die im Falle einer in einer harmonischen Nullmenge punktierten geschlossenen Fläche von positivem Geschlecht mit dem Körper der rationalen Funktionen bzw. mit der Klasse der Abelschen Kovarianten der geschlossenen Fläche zusammenfallen. $\varphi\in\Phi$ wenn $f=\varphi/\psi\in\mathfrak{F}$ und das Quadratintegral $\|\psi\|^2<\infty$ ist. Es ergibt sich noch eine Einschränkung für die a-Stellen einer Funktion $f\in\mathfrak{F}$, $f\not\equiv a$.

K. Strebel (Fribourg)

107:

Ahlfors, Lars V. The method of orthogonal decomposition for differentials on open Riemann surfaces. Ann. Acad. Sci. Fenn. Ser. A. I, no. 249/7 (1958), 15 pp.

Let Γ be the space of square integrable differentials on an open Riemann surface W. The author considers various orthogonal decompositions in the space Γ which are useful in function theory. Some idea of the questions dealt with can be seen from the following list of relevant subspaces of Γ: the subspace Γ_c of closed differentials; the subspace $\Gamma_h=\Gamma_c\cap\Gamma_c^*$, the subspace of harmonic differentials; the subspace Γ_e of exact differentials; Γ_{co}, the closure of the subspace of closed differentials with compact carriers; Γ_{eo}, the closure of the subspace of differentials of functions with compact carriers; and the subspace Γ_{se} of semi-exact differentials, i.e., of differentials which are closed and have zero periods around all dividing cycles.

The space Γ_{hm} is defined by the orthogonal decomposition $\Gamma_h=\Gamma_{hse}+\Gamma_{hm}^*$. It is shown that $\Gamma_{hm}\subset\Gamma_{hc}\cap\Gamma_{ho}$ with equality for finite Riemann surfaces. It is also shown that each differential in Γ_{hm} is the limit in norm of a suitable sequence of harmonic measures on an exhaustion of W. Considerable use is made of Schottky differentials on a finite Riemann surface.

The remainder of the paper deals with the application of the method of orthogonal decomposition to the case of differentials with prescribed singularities and periods, and with a generalization of Abel's theorem.

H. L. Royden (Zürich)

108:

Heins, Maurice. Some remarks concerning parabolic Riemann surfaces. J. Math. Pures Appl. (9) 36 (1957), 305–312.

The author has defined [Ann. of Math. (2) 62 (1955), 418–446; MR 17, 726] a generalization of the Nevanlinna characteristic function to maps of one parabolic surface into another. In this paper he shows that this characteristic function of a mapping f of one parabolic Riemann surface F into a second parabolic surface G is independent of the parameters by which it is defined (Nevanlinna's first fundamental theorem), and that, if this characteristic function does not grow too rapidly, then the mapping f is of constant (finite) valence on G except possibly on a set of capacity zero.

H. L. Royden (Zürich)

109:

Wermer, John. Function rings and Riemann surfaces. Ann. of Math. (2) 67 (1958), 45–71.

Let C be the algebra of all continuous complex functions on the unit circle Γ. Choose f and ϕ in C such that (a) the pair (f, ϕ) separates points on Γ, (b) f and ϕ are analytic on Γ, (c) $\phi'(u)\neq 0$ on Γ, and (d) the smallest closed subalgebra A of C which contains f, ϕ, and the constants, is a proper subalgebra of C. Under these assumptions, the author proves the following interesting theorem: There exists a Riemann surface $\mathfrak{F}$ and a simple closed curve $\dot\gamma$ on $\mathfrak{F}$ with the following properties: $\dot\gamma$ bounds a region $\mathfrak{D}$ on $\mathfrak{F}$ with $\mathfrak{D}\cup\dot\gamma$ compact, and there exists a homeomorphism χ of γ on Γ such that for each g in A the function $\hat g$ defined on $\dot\gamma$ by $\hat g(t)=g(\chi(t))$ has a continuous extension to $\mathfrak{D}\cup\gamma$ which is analytic in $\mathfrak{D}$.

It is first shown that there is no loss of generality in assuming that the curve $\gamma=\phi(\Gamma)$ has only finitely many multiple points. The objects $\mathfrak{F}$, $\dot\gamma$, $\mathfrak{D}$, χ are constructed as follows: For some fixed u_0 on Γ such that $\phi(u_0)$ is not a multiple point of γ, a function f_0 is defined in a neighborhood of $\phi(u_0)$ by $f_0(\lambda)=f(\phi^{-1}(\lambda))$. The Riemann surface obtained by all possible analytic continuations of the function element f_0 is $\mathfrak{F}$, and $\dot\gamma$ is the set of all places on $\mathfrak{F}$ obtained by continuation along γ. Then $\dot\gamma$ is a simple closed curve, and $\chi(u)$ is defined to be the place on $\dot\gamma$ which corresponds to $u\in\Gamma$ (so that $\chi(u)$ projects on $\phi(u)$).

The proof that the objects so constructed have the asserted properties is long and difficult. One of the principal steps consists in showing that among the finitely many bounded components of the complement of γ there is one, W_0, which has a one-sheet region $\dot W_0$ of $\mathfrak{F}$ lying over it, such that for every g in A there is an analytic function G_0 on W_0 which assumes continuous boundary values $\hat g$ on some subarc of $\dot\gamma$; the map $g\to G_0$ is a homomorphism on A. The region $\mathfrak{D}$ is taken as that component of the complement of $\dot\gamma$ in $\mathfrak{F}$ which contains $\dot W_0$.

W. Rudin (New Haven, Conn.)

110:

Rahman, Q. I. On a theorem of Shah. Publ. Math. Debrecen 5 (1957), 40–43.

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is an entire function of order ρ $(0\leq\rho\leq\infty)$, $M(r)=\max_{|z|=r}|f(z)|$, $\mu(r)$ the term maximum for $|z|=r$, $\nu(r)$ its rank, then

$$(1)\quad \liminf_{r\to\infty}\frac{\log\mu(r)}{\nu(r)}\leq\frac{1}{\rho}\leq\frac{1}{\lambda}\leq\limsup_{r\to\infty}\frac{\log\mu(r)}{\nu(r)},$$

where $\lambda=\liminf_{r\to\infty}(\log\log M(r))/\log r$.

For $\rho=0$, a strengthening of (1) is proved: If $f(z)$ is an entire function of order 0, and not constant, then

$$\limsup_{r\to\infty}\frac{\int_{\Delta}^{r}t^{-1}L(t)\nu(t)dt}{\int_{r}^{\infty}t^{-2}L(t)\nu(t)dt}=\infty,$$

with suitable $\Delta>0$, and $L(t)$ any continuous nondecreasing function of t such that $\log L(t)=o(\log t)$.

Another strengthening of (1) is: If $f(z)$ is of lower order λ $(0\leqq\lambda\leqq\infty)$, then

$$\frac{1}{\lambda}\leqq\limsup_{r\to\infty}\frac{\log\mu(r,f)}{\nu(r,f^{(p)})},$$

where $\nu(r,f^{(p)})$ is the rank of the term maximum for $|z|=r$ of the pth derivative of $f(z)$, and p either an arbitrary constant integer or an integer-valued function of r such that

$$p(\nu)=O(\nu/\log\nu)\quad(\nu=\nu(r,f^{(p)})).$$

B. A. Amirà (Jerusalem)

111:

Gol'dberg, A. A. An example of an entire function of finite order with a non-asymptotic defective value. Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 191–194. (Russian)

Hayman [J. London Math. Soc. **28** (1953), 369–376; MR **14**, 965] constructed an entire function of infinite order with a defective value that is not asymptotic, and stated that his construction could be modified to yield a function of large finite order with the same property. The author constructs an entire function of order 3/2 with this property and states that the same idea can be used for any order exceeding 1. Let

$$f_1(z)=\prod(1+z^{2\nu^{-4/3}}),\quad f_2(z)=\prod\phi_n[(-1)^n z],$$

where $\phi_n(z)=(1+iz/r_n)^{\lambda_n}\exp(-\lambda_n iz/r_n)$,

$$r_n=2^{2\cdot8^{n-1}},\quad \lambda_n=8r_n^{3/2}.$$

Then $f(z)=f_1(z)f_2(z)$ has 0 as a defective value that is not asymptotic. *R. P. Boas, Jr.* (Evanston, Ill.)

112:

Ahmad, Mansoor. On exceptional values of entire and meromorphic functions. Compositio Math. **13** (1958), 150–158.

The results are of the following type which goes back to Borel: If f is entire, of growth A, then there is at most one entire f_1 of growth $<A$ such that $f-f_1$ is, in some sense, exceptional. Thus it is shown that (i) if f is of finite order $\rho>0$, there is at most one f_1 of order $<\rho$ such that $\liminf T(r,f)/n(r,f-f_1)>2/\rho$; similarly if f has $\rho=\infty$, and more precisely, finite kth order but infinite $(k-1)$th lower order, while f_1 is of finite $(k-1)$th order; (ii) if f is of zero order, there is at most one polynomial f_1 such that $\liminf\log M(r)/N(r,f-f_1)>6$. Generalizations are given for meromorphic functions; other results deal with simple zeros of functions with positive deficiency sum.

J. Korevaar (Madison, Wis.)

113:

Ahmad, Mansoor. On entire functions of infinite order. Compositio Math. **13** (1958), 159–172.

The author studies entire functions f of infinite order which have finite kth order $\rho_k=\limsup l_k T(r)/\log r$ and infinite $(k-1)$th lower order. For such functions an inequality is proved relating pth order, maximum modulus $M(r)$ and index of maximum term $\nu(r)$: $\liminf l_1M(r)l_2M(r)\cdots l_pM(r)/\nu(r)\leqq1/\rho_p$. It is also shown that $\liminf T(r)l_1T(r)\cdots l_{p-1}T(r)/n(r,f-f_1)>2/\rho_p$ for at most one entire f_1 of finite $(k-1)$th order [compare the preceding review]. The type of a function of finite kth order is defined as $T_k=\limsup l_kM(r)/r^{\rho_k}$. The author gives an expression in terms of the coefficients of the power series, and proves a result on the type of a canonical product of finite kth order.

J. Korevaar (Madison, Wis.)

114:

Turmarkin, G. C.; and Havinson, S. Ya. On the definition of analytic functions of class E_p in multiply connected domains. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 201–206. (Russian)

In question are domains of finite connectivity whose boundary elements may not be rectifiable. It is proposed to eliminate the condition of bounded length applied to the sequence of curves approaching the boundary in previous discussions. [Penez, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 240–243; MR **15**, 784; Havinson, Mat. Sb. N.S. **36(78)**(1955), 445–478; MR **17**, 247; Privaloff, I. I., Randeigenschaften analytischer Funktionen, VEB Deutscher Verlag der Wissenschaft, Berlin, 1956; Boundary properties of analytic functions, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **18**, 727; **13**, 926.]

Among the results obtained are: $f(z)$ belongs to $E_p(G)$ if and only if $|f\{\varphi(w)\}|^p|\varphi'(w)|$ possesses a harmonic majorant in a canonical circular domain (w-plane) represented conformally on G by $z=\varphi(w)$.

For each class $E_p(G)$ there exists a sequence of analytic curves Γ_n (independent of f) converging to the frontier of G, such that $\int_{\Gamma_n}|f(z)|^p|dz|<C(f)$ for each $f(z)\in E_p(G)$. [See also W. Rudin, Proc. Amer. Math. Soc. **6** (1955), 202–204; MR **16**, 810.] *A. J. Macintyre* (Cincinnati, Ohio)

115:

Singh, Vikramaditya. Interior variations and some extremal problems for certain classes of univalent functions. Pacific J. Math. **7** (1957), 1485–1504.

The author solves the following problems: For all bounded univalent functions f in $|z|<1$ normalized so that $f(0)=0$, $|f(z)|\leqq1$, and at a fixed point ζ such that $|\zeta|<1$, $f(\zeta)=\omega$, find the functions which maximize or minimize $|f'(\zeta)|$. For all bounded univalent functions f in $|z|<1$ which are real on the real axis and normalized so that $f(0)=0$, $|f(z)|\leqq1$, and at a fixed point ζ on the real axis $f(\zeta)=\omega$, find the functions which maximize or minimize $f(\eta)$ for real η. In both cases, the extremal functions map $|z|<1$ onto a radially slit unit circle.

G. Springer (Lawrence, Kans.)

116:

Jacobsthal, Ernst. Über die Multiplizität der Fixpunkte einer analytischen Funktion und ihrer Iterierten. Math. Ann. **134** (1957), 134–139.

Three theorems are established on the multiplicity of fixed points of the iterates of analytic functions. These results are extensions of the author's earlier results on polynomials [Math. Z. **63** (1955), 243–276; MR **17**, 574]. The author calls attention to the fact that only after the the publication of that earlier work had he become aware that some of his results had been established earlier by H. D. Block and H. P. Thielman [Quart. J. Math., Oxford Ser. (2) **2** (1951), 241–243; MR **13**, 552].

H. P. Thielman (Ames, Iowa)

117:

Kabaila, V. On interpolation of functions of class H_δ. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 181–188. (Russian)

The problem considered is: Given a sequence of points $\{\lambda_k\}$ in the unit circle and a certain sequence of complex numbers $\{c_k\}$, under what conditions is it possible to construct a function of class H_δ, i.e., analytic in the unit circle and satisfying

$$\lim_{\rho\to1}(2\pi)^{-1}\int_0^{2\pi}|f(\rho e^{i\theta})|^\delta d\theta<\infty\quad(\delta>0),$$

such that $f(\lambda_k)=c_k$ $(k=1, 2, \cdots)$. The condition

$$\sum_{k=1}^{\infty}(1-|\lambda_k|)<\infty \tag{1.2}$$

is necessary but not sufficient.

A typical theorem proved is: For any $\delta\leqq 1$, there exists such a function of class H_δ provided (1.2) is satisfied and

$$\sum_{k=1}^{\infty}\left|\frac{c_k}{b_k(\lambda_k)}\right|^{\delta}(1-|\lambda_k|)^{1-\varepsilon}<\infty \tag{2.1}$$

for some $\varepsilon>0$, where $b_n(z)$ is the Blaschke function

$$\prod_{k=1,k\neq n}^{\infty}\frac{|\lambda_k|}{\lambda_k}\frac{\lambda_k-z}{1-\bar{\lambda}_k z}.$$

It is also shown that if such conditions are relaxed in various ways, then it is possible to find sequences $\{c_k\}$ and $\{\lambda_k\}$ satisfying these conditions and such that there exists no function of class H_δ taking the values c_k at λ_k.

S. Macintyre (Aberdeen)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 157, 227.

118:

Kaizuka, Tetsu; and Michiwaki, Yoshimasa. On some property of bounded analytic transformations. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 137–143.

Let Z be the vector $\begin{pmatrix}z_1\\ \vdots\\ z_n\end{pmatrix}$ and let W be a similar vector with components $w_1, \cdots, w_n$. S. Takahashi [Sûgaku 6 (1955), no. 4] has obtained the following result: If $W=W(Z)=Z+A_2Z^2+A_3Z^3+\cdots$ is regular and if $|W(Z)|<M$ in the hypersphere $|Z|<1$, then i) $W(Z)$ is schlicht in $|Z|<n$ and ii) the image domain of $|Z|<1$ by the mapping $W=W(Z)$ has a schlicht hypersphere the radius of which is R, where

$$r=1/[n(n+1)(4nM+1)],\quad R=1/[2n(n+1)(4nM+1)].$$

It is known that these results are not the best possible; the authors point out, for example, that I. Ono [in an unpublished paper] has obtained the more precise value $r=1/[2(n+1)M]$. In the present paper the authors obtain values which do not depend upon n. In their main result they give $r=2\cdot 3^{\frac{1}{2}}/(9M)$ and $R=3^{\frac{1}{2}}/(9M)$.

W. T. Martin (Cambridge, Mass.)

119:

Ozaki, Shigeo; Kashiwagi, Sadao; and Tsuboi, Teruo. Note on bounded analytic transformations. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 144–148.

Using the notation of the previous paper [# 118 above] the authors show that R may be taken to be

$$1/[3M\sqrt{(3+(M^2-1)^{-1})}].$$

W. T. Martin (Cambridge, Mass.)

120:

Lowdenslager, D. B. Potential theory and a generalized Jensen-Nevanlinna formula for functions of several complex variables. J. Math. Mech. **7** (1958), 207–218.

On considère le domaine homogène symétrique D de $C^n(w_k)$ défini par $|\sum_1^n w_k^2|^2-2\sum_1^n|w_k|^2+1>0$, $1-|\sum w_k^2|^2>0$, qui admet une application analytique complexe sur le tube D' défini par $z_k=x_k+iy_k$, $-\infty<x_k<+\infty$, $1\leqq k\leqq n$, $y_n>(y_1^2+\cdots+y_{n-1}^2)^{\frac{1}{2}}$; D et D' sont considérés comme espaces pourvus de leur métrique de Bergmann $ds^2=\sum g_{ij}dz_i d\bar{z}_j$ (on la calcule explicitement pour D'); on pose $\Delta=\sum g^{ij}(\partial^2/\partial z_i\partial\bar{z}_j)$. On appelle B la partie de la frontière de D où les fonctions holomorphes sur $\bar{D}$ atteignent le maximum du module; $\mathfrak{F}(D)$ désigne l'ensemble des fonctions de classe (C^2) vérifiant $\Delta f=0$ sur D, sauf peut-être aux points de B où l'on exige la continuité. On établit: si u est une fonction continue sur B, il existe une fonction unique $f\in\mathfrak{F}(D)$ qui coincide avec u sur B. Si f est plurisousharmonique dans un ouvert contenant $\bar{D}$, et si $h\in\mathfrak{F}(D)$, $f\leqq h$ sur B entraine $f\leqq h$ dans D. Si G est le groupe d'automorphismes de D laissant invariante l'origine, dg sa mesure de Haar, $\int_G V(gz)dg$ est plurisousharmonique, si $V(z)$ l'est dans D. On donne une représentation:

$$\log|F(w)|+J(w)=\int_B\log|F(b)|P(w,b)db$$

de $\log|F|$, pour F holomorphe sur $\bar{D}$, $w\in D$; $J(w)\geqq 0$; on indique la possibilité de calculer explicitement le terme $P(w, b)$ par référence à D' pour lequel on utilise un résultat L. K. Hua [Harmonic analysis of the classical domains$\cdots$, 1956]; si $J(w_0)=0$ en un point $w_0\in D$, on a $J\equiv 0$ et $f\neq 0$ dans D; la formule demeure valable si on substitue à $\log|F(z)|$ une fonction $V(z)$ plurisousharmonique sur $\bar{D}$.

P. Lelong (Paris)

121:

Fichera, Gaetano. Caratterizzazione della traccia, sulla frontiera di un campo, di una funzione analitica di più variabili complesse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 706–715.

Sia A un campo limitato dello $S_{2n}(x_1, y_1, \cdots, x_n, y_n)$ ove si rappresentano le n variabili complesse $z_k=x_k+iy_k$, Σ sia la frontiera di A che si suppone connessa. È noto il risultato di Severi [gli stessi Rend. (6) **13** (1931), 795–804] secondo cui, nell'ipotesi che Σ sia di classe ω, la traccia w su Σ di una funzione $W(z_1, \cdots, z_n)$ analitica in $A\cup\Sigma$ è caratterizzata, per $n>1$, dalle sole condizioni differenziali: $dw\wedge dz_1\wedge dz_2\wedge\cdots\wedge dz_n=0$. Nel lavoro che si recensisce viene stabilito il teorema di Severi nella sola ipotesi che Σ sia di classe 1 e W sia analitica in A e continua su Σ. Anzi, tale risultato viene subordinato ad altro più generale in cui le indicate condizioni differenziali sono sostituite da condizioni integrali e la funzione traccia ha un significato generalizzato. La dimostrazione, che si svolge nell'ambito della teoria delle funzioni di variabili reali, non fa uso del teorema sul prolungamento di Hartogs, onde ne scende anche una nuova prova di questo.

E. Martinelli (Roma)

122:

Kohn, J. J. A boundary condition for the vanishing of n holomorphic functions in complex n-space. Proc. Amer. Math. Soc. **9** (1958), 175–177.

If $f_1(z_1, \cdots, z_n), f_2, \cdots, f_n$ are holomorphic in $|z_1|^2+\cdots+|z_n|^2<1$, have continuous boundary values on $|z_1|^2+\cdots+|z_n|^2=1$, and if for the latter $\bar{z}_1f_1+\cdots+\bar{z}_nf_n=0$, then $f_1, \cdots, f_n$ are identically zero.

{Reviewer's remark. Many simple proofs and generalizations can be given. For instance, the unit sphere can be replaced by any bounded circular domain.}

S. Bochner (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 185, 186.

123:

***Eagle, Albert. The elliptic functions as they should be: an account, with applications, of the functions in a new canonical form.** Galloway and Porter, Ltd., Cambridge, England, 1958. xxviii+510 pp. 45 s.

"This book effects the second major revolution in the history of the Elliptic Functions. The first, and the one of more fundamental importance, was effected by Abel···" (From the author's preface). The book contains a detailed presentation of elliptic functions and integrals, and of some of their applications, in a new standardization and a new notation.

Contents: Preface. Symbols and notation. Introduction. I. Twenty elliptic functions and their forty trig series. II. Derivatives and squares of the elliptic functions. III. The theta and the peeta functions. IV. The addition theorems. V. Elliptic integrals of the first kind. VI. Elliptic integrals of the second kind. VII. Elliptic integrals of the third and fourth kinds. VIII. The kay and the eee functions. IX. The Weierstrassian poleless functions. X. Applications of elliptic functions to geometry. XI. Applications of elliptic functions to mechanics. XII. Introduction to conformal representation. XIII. The elliptic functions in conformal representation. Miscellaneous examples. XIV. Remarks and conclusion. Supplements. Index. Contents of Chapters X to XIII.

Numerical tables and tables of formulae are given in the text, and most of the chapters are equipped with appendices. The author writes in an unusual style, and some readers may find his presentation unconvincing. Many of the notations used in the book are also unconventional.

A. Erdélyi (Pasadena, Calif.)

124:

Fempl, S. Sur une réduction de l'intégrale elliptique normale complète de III espèce. Srpska Akad. Nauka. Zb. Rad. **55,** Mat. Inst. **6** (1957), 73–76. (Serbo-Croatian. French summary)

By using the results of his previous paper [same Zb. **50**, Mat. Inst. **5** (1956) 61–116; MR **19**, 739], the author derives the necessary conditions for the new reduction of the complete normal elliptic integral of the 3rd class only on the complete integrals of the 1st class with same module. The proof is based on the known Legendre's formula.

D. P. Rašković (Belgrade)

125:

Spiegel, Murray R. The Dirac delta function and the summation of certain Bessel series. J. Math. Phys. **36** (1958), 378–380.

The purpose of this paper is to present an interesting heuristic method for directly obtaining the sums of the series

$$\sum_{m=1}^{\infty} \begin{matrix} J_0 \\ Y_0 \end{matrix}(mx), \quad \sum_{m=1}^{\infty} (-1)^m \begin{matrix} J_0 \\ Y_0 \end{matrix}(mx)$$

by means of a procedure involving application of the Dirac delta function $\delta(x)$. Actually, the author shows how to evaluate the series

$$A_k=\sum_{m=1}^{\infty}(\cos 2\pi km)J_0(mx), \quad B_k=\sum_{m=1}^{\infty}(\cos 2\pi km)Y_0(mx) \qquad (0\leqq k<1),$$

of which the above mentioned series are special cases with $k=0$ and $\frac{1}{2}$ respectively. The formal expansion of $\delta(x)$ in a Fourier series leads to

$$\pi\delta(x)=\tfrac{1}{2}+\sum_{m=1}^{\infty}\cos mx \quad (-\pi<x\leqq+\pi).$$

A simple periodic extension shows

$$\pi D(x)\equiv\pi\sum_{p=-\infty}^{\infty}\delta(x-2p\pi)=\tfrac{1}{2}+\sum_{m=1}^{\infty}\cos mx$$

(p integral, no restriction on x). From

$$J_0(x)=\pi^{-1}\int_0^{\pi}\cos(x\cos\theta)d\theta,$$

the author derives

$$A_k=\sum_p[x^2-4(p-k)^2\pi^2]^{-\frac{1}{2}}+\sum_p[x^2-4(p+k)^2\pi^2]^{-\frac{1}{2}}-\tfrac{1}{2},$$

where p assumes all integers for which the terms in the series are real. Special case: $k=0$:

$$\sum_{m=1}^{\infty}J_0(mx)=\sum_p(x^2-4p^2\pi^2)^{-\frac{1}{2}}-\tfrac{1}{2} \quad (|p|<|x|/2\pi).$$

The results for the series in Y_0 are less simple. They lead to

$$\pi B_k=\lim_{M\to\infty}\int_x^{2M\pi}(u^2-x^2)^{-\frac{1}{2}}[1-\pi D(2\pi k+u)-\pi D(2\pi k-u)]du$$

(M positive integer).

Special case: $k=0$:

$$\pi B_0=2\pi\sum_{p=[x/2\pi]+2}^{\infty}\left(\frac{1}{2p\pi}-\frac{1}{(4p^2\pi^2-x^2)^{\frac{1}{2}}}\right)-\gamma+\sum_{p=1}^{[x/2p\pi]+1}\frac{1}{p}-\ln\left(\frac{x}{4\pi}\right),$$

where γ=Euler constant.

S. C. van Veen (Delft)

126:

Pinney, Edmund. Some discontinuous Bessel integrals. J. Math. Phys. **36** (1958), 362–370.

The author studies the properties of integrals having infinitely many periodically spaced non-null intervals, all of the same length, separating periodically spaced null intervals, also all of the same length. They contain the well-known results:

$$\int_{-\infty}^{+\infty}e^{isy}J_n(bs)ds=\begin{cases}(2/b)i^nT_n(\sin\theta)/\cos\theta, & y^2<b^2,\\ 0, & y^2>b^2,\end{cases}$$

$$\int_{-\infty}^{+\infty}\frac{e^{isy}J_n(bs)ds}{s}=\begin{cases}(1/n)i^{n-1}U_n(\sin\theta), & y^2<b^2,\\ 0, & y^2>b^2,\end{cases}$$

$n\equiv$non-negative integer, $|\theta|<\frac{1}{2}\pi$ with $\sin\theta=y/b$; here T_n and U_n are the Tschebycheff functions of order n [cf. G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944, §13.42; MR 6, 64]. They also contain the finite number of non-null intervals as special cases. They are applicable to boundary value problems involving grids of intervals. The main results are the following.

Theorem I:

$$\int_{-\infty}^{+\infty}e^{isy}J_n(bs)p(s)ds=2i^n\sum_m p_mT_m\left\{\frac{y-m}{b}\right\}/(b^2-(y-m)^2)^{\frac{1}{2}},$$

$$\int_{-\infty}^{+\infty}e^{isy}J_n(bs)p(s)\operatorname{sgn}s\,ds=2b^{-1}i^{n-1}\sum_{m=-\infty}^{\infty}p_mf_n\left(\frac{y-m}{b}\right),$$

$$f_n(h)=\begin{cases}\dfrac{1}{\pi}P\int_0^\infty \dfrac{\cos n\theta}{\theta-h}\,d\theta=\dfrac{U_n(h)}{(1-h^2)^{\frac12}}=\dfrac{T_n'(h)}{n}, & h^2<1,\\[2ex] -(\operatorname{sgn} h)^{n-1}\dfrac{(|h|+(h^2-1)^{\frac12})^{-n}}{(h^2-1)^{\frac12}}, & h^2>1,\end{cases}$$

where the summation is over all integers m for which

$$|(y-m)/b|<1,\ p_m=(2\pi)^{-1}\int_{-\pi}^{+\pi}e^{ims}p(s)ds;$$

$p(s)$ continuous, periodic, of period 2π, such that the integrals converge.

Theorem II:

$$\int_{-\infty}^{+\infty}e^{isy}J_n(bs)q(s)ds/s=$$

$$(i^{n-1}/n)\sum_m q_m U_n\{(y-m)/b\},\ |(y-m)/|b|<1,$$

$$\int_{-\infty}^{+\infty}e^{isy}J_n(bs)q(s)ds/|s|=(2/n)i^n\sum_{m=-\infty}^{+\infty}q_m g_n\{(y-m)/b\},$$

where $q(s)$ is periodic, of period 2π, $L(-\pi,\pi)$, of bounded variation for $|s|<\pi$; $q_m=\pi^{-1}\int_{-\pi}^{+\pi}e^{ims}q(s)ds$; $g_n(h)=T_n(h)$ for $h^2<1$, $=(\operatorname{sgn} h)^n(|h|+(h^2-1)^{\frac12})^{-n}$ for $h^2>1$. Another result, not of immediate interest in boundary value problems, Theorem III, which is closely related to Theorem I, is given without proof:

$$\sum_{k=-\infty}^{+\infty}J_n(a+kb)e^{ikv}=$$

$$2i^n\sum_m e^{-i(v-2m\pi)a/b}T_n\{(v-2m\pi)/b\}/\{b^2-(v-2m\pi)^2\}^{\frac12},$$

where $n\equiv$non-negative integer, $|(v-2m\pi)/b|<1$.

S. C. van Veen (Delft)

127:

Ragab, F. M. On the product of Legendre and Bessel functions. Proc. Amer. Math. Soc. **9** (1958), 26–31.

Integrals of the form

$$\int_0^\infty \lambda^{\gamma-1}(1+\lambda)^{-\gamma/2}C_\nu[z(1+\lambda)^{\frac12}]\,{}_2F_1(\alpha,\beta;\gamma;-\lambda)d\lambda$$

$$(\Re\gamma>0,\ |\arg z|<\pi),$$

where C_ν is a cylinder function of order ν, are evaluated.

N. D. Kazarinoff (Ann Arbor, Mich.)

128:

Mursi, Z.; and Yacoub, K. R. On a generalization of Airy's integrals. Proc. Math. Phys. Soc. Egypt **5** (1955), no. 3, 97–110 (1957).

The author studies the general solution of the generalised Airy equation $y^{(k)}-xy=0$. The series solution is

$$(1)\quad y=\sum_{p=0}^{k-1}a_p x^p\sum_{r=0}^{\infty}\frac{(k+1)^r}{(kr+r+p)!}\Gamma(r+(p+1)/(k+1))x^{r(k+1)},$$

where a_p is arbitrary and k an integer. Definite integral solutions, obtained by Laplace transformation, are the real and imaginary part of

$$\overset{\nu}{\underset{k}{\mathrm{Ci}}}\,x+i\overset{\nu}{\underset{k}{\mathrm{Si}}}\,x=\int_0^\infty\exp\Big(xt\exp\frac{2\nu\pi i}{k+1}-\frac{t^{k+1}}{k+1}\Big)dt$$

$$+\exp\frac{-2\nu\pi i}{k+1}\int_0^\infty\exp\Big(xt-\frac{t^{k+1}}{k+1}\Big)dt$$

for $\nu=0,\pm1,\pm2,\cdots,\pm\frac12 k$ (k even), $\nu=0,\pm1,\pm2,\cdots,\frac12(k-1),\frac12(k+1)$ (k odd). Expansion in power series gives expressions of the form (1) with

$$a_p=c_p(\nu)+is_p(\nu)=$$

$$(k+1)^{(p-k)/(k+1)}[\exp(-i\beta)+\cos p\beta\sum_{m=0}^{p}{}_pC_m\tan^m\beta\exp\tfrac12 mi\pi],$$

where $\beta=2\pi\nu/(k+1)$.

S. C. van Veen (Delft)

129:

Ivanenko, V. V. Solutions of the Lamé equation from a general point of view. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 42–61. (Ukrainian)

130:

Dörr, Johannes. Mathieusche Funktionen als Eigenfunktionen gewisser Integralgleichungen. Z. Angew. Math. Mech. **38** (1958), 171–175.

Mathieu functions, that is to say 2π-periodic solutions of $d^2\phi/dz^2+(q-k^2\cos^2 z)\phi=0$, satisfy integral equations of the form $\phi(\alpha)=\lambda\int_{-\pi}^{\pi}K(\alpha,\beta)\phi(\beta)d\beta$. The author's aim is to find new nuclei for the latter equation; K must satisfy $(\partial^2/\partial\alpha^2-k^2\cos^2\alpha)K=(\partial^2/\partial\beta^2-k^2\cos^2\beta)K$, and certain conditions of periodicity and differentiability. The main new solution is

$$K_s(\alpha,\beta)=\exp[-\tfrac12 k\{s\cos(\alpha\pm\beta)-s^{-1}\cos(\alpha\mp\beta)\}],$$

where s is an arbitrary parameter; by transformation, and differentiation and integration with respect to s, further nuclei are found which involve Bessel functions, and methods of calculating the corresponding eigenvalues are also indicated.

F. M. Arscott (Aberdeen)

131:

Höhn, Erwin. Numerische Untersuchungen zu den von J. Dörr angegebenen Integralgleichungen. Z. Angew. Math. Mech. **38** (1958), 175–179.

The author shows how to calculate eigenvalues of the integral equations for Mathieu functions given by Dörr [≠ 130 above], taking as examples

$$K(\alpha,\beta)=J_0\{k(\cos\alpha-\cos\beta)\},\ K(\alpha,\beta)=N_0\{k(\cos\alpha-\cos\beta)\}.$$

Graphs and tables are given for the first six eigenvalues of each equation for $k^2=0(1)10(10)100$.

F. M. Arscott (Aberdeen)

132:

Shukla, H. S. Certain transformations of nearly-poised bilateral hypergeometric series. Ganita **7** (1956), 113–121.

The author gives the transformation of a nearly-poised basic bilateral series ${}_5\Psi_5$, in terms of a terminating well-poised ${}_{12}\Psi_{12}$. From this, he deduces a number of special cases, including the sum of the series

$${}_5\Psi_5\left[\begin{matrix}aq^{-N}, & q\sqrt{a}, & -q\sqrt{a}, & dq^N, & q^{-N}\\ q^{N+1}, & \sqrt{a}, & -\sqrt{a}, & aq^{-N+1}/d, & d^2q^{2-N}\end{matrix};q\right]$$

and the corresponding transformation for an ordinary nearly-poised bilateral ${}_5H_5$ series.

He also gives the transformation of a nearly-poised basic ${}_7\Psi_7$ series in terms of another terminating well-poised ${}_{12}\Psi_{12}$, and the corresponding transformation of a nearly-poised ${}_6H_6$ in terms of a well-poised ${}_9H_9$ series.

L. J. Slater (Cambridge, England)

133:

Palamà, Giuseppe. Su alcuni polinomi che generalizzano quelli di Languerre e su altri che generalizzano quelli di Hermite ed i loro associati. Riv. Mat. Univ. Parma **7** (1956), 293–309.

The author continues the study of the generalized Hermite polynomials introduced in an earlier paper [same Riv. **4** (1953), 363–386; MR **16**, 470] and studies also a generalized Laguerre polynomial defined by

$$\Psi_{n,\nu}^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\Psi_1(-n;\nu+1;-\alpha-2n-\tfrac12,\alpha+1;2,x),$$

where Ψ_1 is one of Humbert's confluent hypergeometric functions of two variables.

A. Erdélyi (Pasadena, Calif.)

134:

Gatteschi, Luigi. Limitazione degli errori nelle formule asintotiche per le funzioni speciali. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 83–94.

After reviewing different methods of estimating errors in asymptotic representations of special functions, the author considers the corresponding problem for the zeros of functions. Assume that $f(x, \mu)$ allows an asymptotic representation of the form $f(x, \mu)=g_0(x)+g_1(x)\mu+\varepsilon_2(x)\mu^2$ as $\mu\to 0$, where g_0, g_1, ε_2 may also depend on μ but remain uniformly bounded. Let x^* be a zero of f and x_0 a zero of the first term g_0. Then, as is well known, under appropriate conditions, $x^*=x_0-[g_1(x_0)/g_0'(x_0)]\mu+O(\mu^2)$. The author makes this relation more precise by giving an explicit estimate of the O-term. As an application he derives a simple new inequality for the zeros $\theta_{n,r}$ of $P_n(\cos\theta)$ and the following more precise result

$$\theta_{n,r}=\alpha_{n,r}+\frac{\cot\alpha_{n,r}}{2(2n+1)(2n+3)}+\rho \quad (r=1, 2, \cdots, [\tfrac{1}{2}n]),$$

$$\alpha_{n,r}=\frac{r-\frac{1}{4}}{n+\frac{1}{2}}\pi, \quad |\rho|<\frac{4(\sqrt{2})+9}{8(2n+3)(4r-1)^2}.$$

Walter Gautschi (Washington, D.C.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 184, 301, 668, 797, 798, 799, 803.

135:

***Cunningham, W. J. Introduction to nonlinear analysis.** McGraw-Hill Electrical and Electronic Engineering Series. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. ix+349 pp. $9.50.

This book is on the non-linear ordinary differential equations of engineering interest. The treatment is non-rigorous but the author does indicate the limitations of the various methods considered. The reader is assumed to be familiar with electrical circuits, mechanical vibrations and linear differential equations with constant coefficients. *N. Levinson* (Cambridge, Mass.)

136:

***McLachlan, N. W. Ordinary non-linear differential equations in engineering and physical sciences. 2nd ed.** Oxford University Press, New York, 1956. x+271 pp. $5.60.

[First edition was reviewed in MR 12, 500.] In this edition the treatment of several topics is revised; a long chapter on trajectories and stability criteria in the two dimensional phase-plane is added, as is a short chapter on the third order ordinary differential equation arising in certain cases of boundary layer theory in plane flow.

N. Levinson (Cambridge, Mass.)

137:

Corduneanu, C. Quelques considérations concernant certains systèmes non linéaires d'équations différentielles. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **7** (1956), no. 2, 13–32. (Romanian. Russian and French summaries)

In the first part of this paper, the author investigates the system (*) $dx_i/dt=f_i(t, x_i)+g_i(t, x_1, \cdots, x_n)$ $(i=1, \cdots, n)$ under the following conditions: (i) f_i and g_i are continuous for $t\geqq 0$ and for all real x_i; (ii) $f_i(t, 0)+g_i(t, 0, \cdots, 0)$ are bounded on $t\geqq 0$; (iii) there exist positive numbers m and M so that for each $i=1, \cdots, n$ either $-M\leqq\partial f_i/\partial x_i\leqq -m$ or $m\leqq\partial f_i/\partial x_i\leqq M$; (iv) the g_i satisfy a Lipschitz condition, and the Lipschitz constant is $<m/n$. Under these conditions the author establishes the existence, and some properties, of bounded solutions. He investigates the stability of these solutions, and extends his investigations to the case $-\infty<t<\infty$ and to the case where the x_i are restricted to a bounded region, $\sum|x_i|\leqq a$.

In the second part of the paper he investigates similarly the system obtained by replacing dx_i/dt in (*) by d^2x_i/dt^2.

A. Erdélyi (Pasadena, Calif.)

138:

Corduneanu, C. Systèmes différentiels admettant des solutions bornées. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 2, 107–126. (Romanian. Russian and French summaries)

The author investigates the system $dx_i/dt=f_i(t, x_1, \cdots, x_n)$ $(i=1, \cdots, n)$ under conditions similar to those mentioned in the preceding review. In the present paper he establishes in particular the existence of solutions satisfying certain initial conditions, and the existence of periodic solutions in case the f_i are periodic functions of t. In a typical case, he considers an operator A which, operating on $(u_1, \cdots, u_n)$, results in $(y_1, \cdots, y_n)$, where $dy_i/dt=\pm My_i+f_i(t, u_1, \cdots, u_n)\mp Mu_i$ $(i=1, \cdots, n)$; here the upper signs are to be taken, and y_i is bounded as $t\to+\infty$, if f_i satisfies the second of the inequalities under (iii) of the preceding review, and the lower signs are taken and $y_i(0)=0$ if f_i satisfies the first of those inequalities. The solution $(x_1, \cdots, x_n)$ is a fixed point of this operator.

As in the paper reviewed above, the investigations are extended to second order systems.

A. Erdélyi (Pasadena, Calif.)

139:

Zbornik, Josef. Ein neuartiges Verfahren zur Uniformierung und allgemeinen Lösung von linearen Differentialgleichungen und zur Herleitung ihrer Rekursionsformeln. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **166** (1957), 21–62.

L'auteur considère l'equation

$$(*) \qquad O(y)=E_1+Bx^rE_2=H(x),$$

qu'il écrit en forme symbolique

$$O(y)=(\prod_{i=1}^{m} O_{p_i}+Bx^r\prod_{j=1}^{n} O_{q_j})(y)=H(x).$$

B, r sont constantes, $E_1=0$, $E_2=0$ des équations différentielles d'Euler, $O_\alpha=x^{\alpha+1}d^\alpha x^{-\alpha}/dx$.

La forme ainsi écrite permet de rendre uniforme et résoudre d'une façon systématique les équations de forme (*), de même que d'obtenir l'intégrale générale dans une forme explicite.

Un grand nombre d'équations connues peuvent être considérées comme des cas particuliers de l'équation donnée plus haut, ce que l'auteur démontre d'ailleurs par plusieurs exemples.

{La bibliographie citée n'est pas complète.}

B. S. Popov (Skopje)

140:

Banditch, Ivan. Über eine Differentialgleichung erster Ordnung. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 2, 37–38.

Es wird gezeigt, wie es in einigen Fällen möglich ist die Differentialgleichung der Form $f(x)y^{n-1}y'+\varphi(x)y^n=y'^k$ durch Quadraturen zu lösen. *M. Zlámal* (Brno)

141:

Jastrzębska-Olech, J. Sufficient conditions for all integrals of a stationary system to be determined in the bilaterally unbounded interval. Ann. Polon. Math. **4** (1958), 220–225.

The system considered is of the form (1) $dy/dt=F(y)$ where y is an n-dimensional vector and $F(y)$ is the continuous vector-function for y belonging to a domain of the n-dimensional Euclidean space. The author proves a sufficient condition for every solution of (1) to be defined for $-\infty<t<+\infty$ and deduces from it a strengthening of a theorem of Winograd [see Nemyckiĭ and Stepanov, Qualitative theory of differential equations, 2nd ed., Gostehizdat, Moscow-Leningrad, 1949, p. 28; for a review of the 1st ed. see MR **10**, 612]. *M. Zlámal* (Brno)

142:

Fort, Tomlinson. A problem of Richard Bellman. Proc. Amer. Math. Soc. **9** (1958), 282–286.

Consider the Sturm-Liouville problem $u''+\lambda(f(x)+\varepsilon g(x))=0$, $u(0)=u(1)=0$, where $f(x)$ and $g(x)$ are continuous functions over [0, 1] with positive minima. The author shows by passage to the limit from a corresponding result for linear systems of recurrence relations that the smallest characteristic root, λ_1, regarded as a function of ε, can be analytically continued over any finite region not including the negative real axis. The singularity of this function nearest the origin is the expression $-\min_x f(x)/g(x)$ where x is in [0, 1]. This answers a problem posed by the reviewer [Bull. Amer. Math. Soc. **63** (1957), 59]. *R. Bellman* (Santa Monica, Calif.)

143:

Bellman, Richard. Functional equations in the theory of dynamic programming. VIII. The variation of Green's functions for the one-dimensional case. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 839–841.

Let $K(x, y; a)$ be the Green's function corresponding to the differential equation $u''(x)+q(x)u(x)=v(x)$, $a<x<1$, $u(a)=u(1)=0$. Under appropriate conditions, consideration of the function

$$f(a, c)=\max \int_a^1 [q(x)u^2(x)-u'^2(x)-2v(x)u(x)]dx$$

(under the constraint $u(a)=c$, $u(1)=0$) yields a formula for $\partial K(x, y; a)/\partial a$. *J. Kiefer* (Oxford)

144:

Griščenko, L. Z. Asymptotic behavior of solutions of a differential equation of second order with a coefficient of bounded variation. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 62–70. (Ukrainian)

145:

Vas'kivs'ka, T. G. Asymptotic behavior of solutions of a linear differential equation of second order. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 71–97. (Ukrainian)

146:

Kondrat'ev, V. A. On the oscillatoriness of solutions of linear differential equations of the third and fourth order. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 22–24. (Russian)

The question dealt with refers to the oscillatoriness of the solutions of

(1) $y'''+p(x)y=0$

(2) $y^{(4)}+q(x)y=0$

with $p(x)$, $q(x)$ continuous for $x\geqq x_0$, in the sense that the solutions do or do not have an infinite number of zeros. In the short note seven theorems are stated with few hints of proofs. Somewhat typical and simplest are the following. Th. 1: If $p(x)$ is of fixed sign then between consecutive zeros of a solution of (1) there are at most two of any other solution. Th. 2: Same as Th. 1 for (2) save "at most four" replaces "at most two". Th. 3: If $p(x)$ is of fixed [variable] sign then there does [does not] exist an oscillatory solution of (1). If $p\geqq 0$ [$p\leqq 0$] there exists a positive monotone decreasing [increasing] solution of (1). Th. 4: If $\int^{+\infty} |p|x\,dx$ does not converge then there is a fundamental system of solutions of (1) of which two are oscillatory and the third is not. [References: Švec, Czechoslovak Math. J. **4(79)** (1954), 75–94; Sobol', same Dokl. **61** (1948), 219–222; MR **16**, 476; **10**, 40]. *S. Lefschetz* (Princeton, N.J.)

147:

Krasnosel'skiĭ, M. A. Periodic solutions in the neighbourhood of a singular point of a dynamical system. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 180–183. (Russian)

Consider the system of differential equations $\ddot{x}_i+g_i(x_1\cdots x_n, \dot{x}_1\cdots\dot{x}_n)=0$ $(i=1\cdots n)$, where the functions g_i are all odd as functions of the vector $(x_1\cdots x_n)$. The origin is then a stationary point and we turn our attention to periodic solutions with small amplitude. Let $\mathfrak{R}$ be a set, either countable or uncountable, of such periodic solutions, and let $\mathfrak{R}^*$ be the set of points in $2n$-dimensional phase space which lie on the closed integral curves corresponding to the solutions in $\mathfrak{R}$. Then $\mathfrak{R}$ is said to have the limit period T if the periods of the solutions in $\mathfrak{R}$ approach T as their amplitudes approach zero. And $\mathfrak{R}$ is said to form a continuous branch if $\mathfrak{R}^*$ intersects the boundary of every small enough neighborhood of the origin of phase space. The author states without proof several strange theorems concerning the existence of such sets $\mathfrak{R}$ and their further properties. One theorem, for instance, states this: Consider the matrix C with components $\partial g_i(0\cdots 0)/\partial x_j$, and suppose that $4\pi^2/T_k^2$ is a positive eigenvalue of C with odd multiplicity. Suppose also that if we consider all those other positive eigenvalues of which some integral multiple is $4\pi^2/T_k^2$, then the sum of their multiplicities is even. It follows that there exists a set $\mathfrak{R}_k$ of periodic solutions with limit period T_k and that $\mathfrak{R}_k$ forms a continuous branch. A second theorem states that $\mathfrak{R}_k^*$ is tangent at the origin to a certain subspace of phase space. Yet another theorem concerns a perturbation of this system with a driving force having a given period T in time. Suppose that $4\pi^2/T_k^2$ is a simple eigenvalue of C and that certain further conditions are satisfied. Then if T lies on a specified side of T_k but is not too far away, there exist no less than three periodic solutions of the perturbed system.

These theorems, according to the author, can all be proved using general theorems from functional analysis. *G. Hufford* (Seattle, Wash.)

148:

Santoro, Paolo. Studio qualitativo del sistema $\dot{x}=ax^2+bxy+cy^2+f(x, y)$, $\dot{y}=dx^2+exy+hy^2+g(x, y)$ nell'intorno del punto singolare (0, 0). Boll. Un. Mat. Ital. (3) **12** (1957), 566–590.

Consider the real system in the title where not all the constants a, b, c, d, e, h are zero, $f, g\in C^2$, $f=\rho^3\varphi$, $g=\rho^3\psi$, $\rho=(x^2+y^2)^{\frac{1}{2}}$, $\varphi, \psi\in C^2$; it is assumed that (0, 0) is an isolated singular point. The system without f, g is called a reduced system, and degenerate if the second-degree

terms have a common real linear factor; the main purpose of this paper is to study this degenerate system. The singular point of a degenerate reduced system may be of 14 different (not affinely equivalent) types. For the complete system, the main result is that there always exists an integral curve which tends to the origin in a definite direction. *J. L. Massera* (Montevideo)

149:

Nagumo, Mitio; and Isé, Kusuo. On the normal forms of differential equations in the neighborhood of an equilibrium point. Osaka Math. J. **9** (1957), 221–234.

Consider (1) $\dot{x}=Ax+f(x)$ and (2) $\dot{y}=Ay$, where x, y, f are real m-vectors, A a real constant matrix all of whose characteristic roots λ_i have non-vanishing real parts. There exists a constant $h_0>0$ depending only on the λ_i such that if $f=p+q$, p a polynomial which vanishes at the origin together with its first partial derivatives, q a function of class C^1 such that $q(0)=0$, $|\partial q_i/\partial x_j|\leqq Q\|x\|^h$ for some constant Q and an integer $h>h_0$, and if no relation of the form $\lambda_i=\sum_{j=1}^{m} n_j\lambda_j$, n_j integers $\geqq 0$, $1<\sum n_j<h$, is satisfied, then there exists a change of variables $y=x+u(x)$, $u\in C^1$ in a neighborhood of the origin, $u(0)=0$, $|\partial u_i/\partial x_j|\leqq L\|x\|$, $L>0$ a constant, which transforms (1) into (2). *J. L. Massera* (Montevideo)

150:

Brauer, George. Remarks on a paper by Utz. Proc. Amer. Math. Soc. **9** (1958), 34–36.

If $\alpha(x, y)$ and $\beta(x, y)$ are real functions for which there exists a real constant T such that $\alpha(x, 0)=0$, $\beta(x, y)<0$ for $x<0$ and $y>T$, $\beta(x, y)>0$ for $x>0$ and $y>T$, and if $x(t)$ is a solution of

$$x''+\alpha(x, x')+\beta(x, t)=0$$

valid for all large values of t, then $x(t)$ is either oscillatory in the strong sense or is ultimately bounded. If in addition to these hypotheses, $x(t)$ is known to be ultimately bounded and if

$$\limsup_{y\to\infty}\int_a^y \alpha[x(y), x'(y)]/x'(y)dy<\infty,$$

then $x(t)$ is ultimately increasing or decreasing. Let $c(t)$ and $g(t)$ be differential functions such that $c(t)>0$, $c'(t)\geqq 0$, for $t\geqq T$; $g(0)=0$, $g'(t)\leqq 0$. Then if $x(t)$ is a solution of the equation

$$x''+g(x')+c(t)x=0$$

valid for all large values of t, then either $x(t)$ is oscillatory, $\lim_{t\to\infty} x(t)=\infty$ or $\lim_{t\to\infty} x(t)=-\infty$. These theorems generalize results of the reviewer [Proc. Amer. Math. Soc. **7** (1956), 1047–1048; MR **19**, 141]. *W. R. Utz* (Columbia, Mo.)

151:

Ziemba, Stefan. Free vibration of systems of one degree of freedom with non-linear elastic characteristic and non-linear viscous-type damping. Arch. Mech. Stos. **10** (1958), 163–193. (Polish and Russian summaries)

Consider the equation (1) $x''+\Phi(x')+F(x)=0$, where $\Phi(w)$, $F(w)$ are analytic in $(-\infty, +\infty)$, odd functions of w, zero when $w=0$, strictly monotone increasing in w and have the same sign as w. The author discusses general properties of the zeros of the solution of (1) and its limiting behavior as $t\to\infty$. Very detailed information is given about the oscillatory character of the solutions of (1) for all possible combinations of hard or soft elastic and hard or soft damping characteristics. Also, the solution of (1) is given explicitly for some particular functions $F(x)$, $\Phi(x')$. *J. K. Hale* (Baltimore, Md.)

152:

Skačkov, B. N. Questions of stability in the large and regulation qualities for certain systems of differential equations. II. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 19, 35–46. (Russian. English summary)

After a linear transformation of the basic variables the system under consideration may be given the form

$$\begin{aligned}\dot{\eta}_1&=r_1\eta_1+n_1\xi\\ \dot{\eta}_2&=r_2\eta_2+n_2\xi\\ \dot{\xi}&=f(\eta_1, \eta_2, \xi),\end{aligned}\tag{1}$$

where the n_i and r_i are real constants, the $r_i<0$, f is continuous, vanishes on the plane $\sigma=p_1\eta_1+p_2\eta_2-\xi=0$ and is such that $\sigma\cdot f(\eta_1, \eta_2, \xi)>0$.

In Part I [same Vestnik **13** (1957), no. 3, 67–80; MR **19**, 416] the author showed that if

$$1+\frac{|n_1p_1|}{r_1}+\frac{|n_2p_2|}{r_2}>0\tag{2}$$

then (1) is stable in the large. Assuming (2) he now obtains estimates for η_1, η_2, ξ in terms of their initial values $\eta_1^{(0)}$, $\eta_2^{(0)}$, $\xi^{(0)}$. The same subdivision in a large number of special cases is required as in the previous paper. [Additional reference: Škačkov, same Vestnik **12** (1957), no. 1, 46–56, 208; MR **19**, 235.] *S. Lefschetz* (Princeton, N.J.)

153a:

Aĭzerman, M. A.; and Gantmaher, F. R. Stability by means of a linear approximation of periodic solutions of a system of differential equations with discontinuous right-hand sides. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 527–530. (Russian)

153b:

Aĭzerman, M. A.; and Gantmaher, F. R. Stability by means of a linear approximation of periodic solutions of a system of differential equations with discontinuous right-hand sides. Prikl. Mat. Meh. **21** (1957), 658–669. (Russian)

Let the n-vector system

$$\dot{z}=f(z, t),\tag{1}$$

where f has the period τ in t possess a periodic solution $z^0(t)$ of period τ. The stability of this solution was investigated by Liapunov by reducing it to that of the origin for the linear variation equation in case its characteristic roots have non-zero real parts (non-critical case). The same general problem is treated by the present authors when $f(z, t)$ has discontinuities on the curve $z=z^0(t)$. These discontinuities occur as follows. In the $(n+1)$-space of (z, t) let C be a curvilinear cylinder whose axis is $z=z^0(t)$. Let the infinite sequence of varieties of discontinuity $F_\alpha(z, t)=0$ intersect the axis in points M_α at times t_α and the cylinder C in regions H_α (between F_α and $F_{\alpha+1}$) so that the trajectory $z^0(t)$ always crosses the F_α from their negative to their positive sides. The assumptions are: (a) f has the period τ, and in each H_α it is of class C^1, with finite jumps at the F_α; (b) F_α is smooth at M_α; (c) on $z^0(t)$ the derivatives $\dot{F}_\alpha^+$, $\dot{F}_\alpha^-\neq 0$ and of the same signs; (d) under a displacement of τ along the axis the family $\{F_\alpha\}$ is unchanged.

Consider now the variation system

$$\dot{x}=\left[\frac{\partial f}{\partial z}\right]_{z^0}\cdot x.\tag{2}$$

Suppose that in $[t_{\alpha-1}+0, t_\alpha-0]$ the solution $x(t)$ satisfies (2), but at time t_α we have one of the equivalent relations

$$(3)\qquad \begin{cases} x^+-x^-=(h\cdot x^-)\cdot\xi \\ x^+-x^-=(h\cdot x^+)\cdot\xi \end{cases}$$

where ξ is the (vector) jump of f at M_α upon passing from the $-$ to the $+$ side of F_α, and

$$\left\{h^\pm=(\dot F_\alpha^\pm)\cdot\frac{\partial F_\alpha}{\partial z}\right\}_{M_\alpha}.$$

Let (2) and (3) together be described as the linear approximations (l.a.) to (1). Th. 1: If the origin is asymptotically stable for the l.a. then so is $z^0(t)$.

One may determine the characteristic exponents (c.e.) as in the continuous case. Th. 2: If one of the c.e. >1 in absolute value then $z^0(t)$ is unstable.

S. Lefschetz (Princeton, N.J.)

154:

Tkačev, V. F. New sufficient conditions for stability, semistability and instability of the limit cycle of the equation $dy/dx=P(x, y)/Q(x, y)$. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 564–567. (Russian)

Let a planar system possess a limit-cycle L in the neighborhood of which the system may be written

$$(1)\qquad \frac{dn}{ds}=F(s, n),$$

where s is the arc length on L and n the normal distance from L. It is supposed that F has all the partials required below and also that s, n may be taken as coordinates in a suitable neighborhood of L. According to Papuš [Uspeh. Mat. Nauk **7** (1952), no. 4(50), 165–168; MR **14**, 472] L is said to be regular if in (1) F has fixed sign near L on one and the other side of L (the two signs need not be equal). Generalizing results of Papuš [loc. cit.] and of Čečik [ibid. **10** (1955), no. 1(63), 183–187; MR **17**, 37], the author gives explicit sufficient analytic conditions on the partials of F for stability, semistability, instability. In the regular case for stability [instability] they simplify to this: k odd:

$$F_n^{(i)}|_{n=0}=0,\ i<k;\ F_n^{(k)}|_{n=0}<0\ [>0];$$

k even and instability: same except that the last is

$$F_n^{(k)}|_{n=0}\neq 0.$$

The conditions just given are likewise regularity conditions for the limit-cycle.

S. Lefschetz (Princeton, N.J.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 97, 438, 439, 637, 656.

155:

Sokolov, P. V. The complete integral and its envelope; the relation between the normal and the characteristic cones of a differential equation. Penzen. Gos. Ped. Inst. V. G. Belin. Uč. Zap. **4** (1956), 61–74. (Russian)

156:

Koval', P. I. On stability of an approximate solution of a mixed problem for certain differential equations. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 32–41. (Ukrainian)

157:

Leray, Jean. Uniformisation de la solution du problème linéaire analytique de Cauchy près de la variété qui porte les données de Cauchy. C. R. Acad. Sci. Paris **245** (1957), 1483–1488.

The author gives a far reaching extension of the Cauchy-Kowalewski theory so as to cover the case when the initial surface S may have characteristic points (but the case where S is completely characteristic is excluded).

Let $a(x, \partial/\partial x)$ be a holomorphic differential operator of degree m on the complex analytic variety X of dimension l. Let S be a regular subvariety of dimension $l-1$ which is defined locally by an equation $s(x)=0$, s holomorphic, $(\partial s/\partial x)\neq 0$. Let $b(x, \partial/\partial x)$ be a holomorphic differential operator of degree 1 defined in a neighborhood of S and for which S has no characteristic points. Let $v(x)$ and $w_j(x)$ be given holomorphic functions defined in a neighborhood of S ($j=0, 1, \cdots, m-1$). Cauchy's problem consists of studying the analytic continuation properties, near S, of the function $u(x)$ which is defined and holomorphic in a neighborhood in X of the non-characteristic points of S and satisfies: $a(x, \partial/\partial x)u(x)=v(x)$; $u(x)=w_0(x)$, $b(x, \partial/\partial x)u(x)=w_1(x)$, $\cdots$, $[b(x, \partial/\partial x)]^{m-1}u(x)=w_{m-1}(x)$ for $x\in S$.

A neighborhood of S above X is defined by (1) a complex analytic variety Φ of dimension l; (2) a regular subvariety Σ of Φ of dimension $l-1$; (3) a holomorphic map $x(\varphi)$ of Φ into X whose restriction to Σ is an analytic homeomorphism onto S, and such that the functional determinant $D(x)/D(\varphi)$ is not identically zero on Σ. We call Δ the analytic subvariety of Σ defined by $D(x)/D(\varphi)=0$; we identify S with Σ and say that Φ is ramified above Δ.

The main result of the author is that he can construct a neighborhood Φ of S above X so that $u(x)$ is holomorphic on Φ.

Finally, the author shows that under certain conditions, Φ is algebroid, that is, there are only a finite number of points in Φ which project on a given point of X.

L. Ehrenpreis (Waltham, Mass.)

158:

Leray, Jean. La solution unitaire d'un opérateur différentiel linéaire et analytique. C. R. Acad. Sci. Paris **245** (1957), 2146–2152.

Notations: X is a domain in complex affine l-space; x, y are points in X; ξ, η are linear functions on X; $\xi\cdot x=\xi_0+\xi_1x_1+\cdots+\xi_lx_l$; Ξ is the complex affine $(l+1)$-space; and ξ^* is the hyperplane in X: $\xi\cdot x=0$.

Let $a(x, \xi)$ be a polynomial in ξ independent of ξ_0; call $b(\xi, x)=a(x, \xi)$, $a^*(\xi, x)=a(x, -\xi)$. Let $U(\xi, y)$ be the solution of the Cauchy problem of order m: $a(y, \partial/\partial y)\,U(\xi, y)=1$, $U(\xi, y)$ has a zero of order $m-1$ on $\xi\cdot y=0$. Clearly, $U(\xi, y)$ is a function of (ξ^*, y).

We may now apply the method of the paper reviewed above to uniformize $U(\xi, y)$ considered as a function on the product space $\Xi\times X$. Actually, in this case, the results are simpler than in the previous paper.

Suppose now that $a(x, \xi)$ is a polynomial in (x, ξ). Call n the smallest integer such that $x_0^n a(x/x_0, x_0\xi)=A(x_0, x_1, \cdots, x_l, \xi_1, \cdots, \xi_l)$ is a polynomial in (x_0, x, ξ). The Laplace transform of $a(x, \partial/\partial x)$ is the operator of order $n+m$: $A(-\partial/\partial\xi, \xi)$. The author states that $a(x, \partial/\partial x)$ and its Laplace transform have the same bicharacteristics.

For r an integer ≤ 0 define $U_r(\xi, y)$ as the solution of the Cauchy problem: $'y, \partial/\partial y)U_r(\xi, y)=(-\xi\cdot y)^{-r}/(-r)!$ and $U_r(\xi, y)$ has a zero of order $m-r$ on $\xi\cdot y=0$. We have $U_{r+1}(\xi, y)=-(\partial/\partial\xi_0)U_r(\xi, y)$. We use this relation to

define U_r for $r>0$. Similarly we define $U_r^*(\xi, y)$ for $a^*(\partial/\partial x, x)$. The author states that $U_{-n}^*(\xi, y)$ is the unitary solution of $A(-\partial/\partial\xi, \xi)$.

The author shows how to compute U_m and U_m^* when $a(x, \partial/\partial x)$ is linear in x and homogeneous in $\partial/\partial x$.

Using the author's functional transformation J [same C. R. **242** (1956), 953–959; MR **17**, 1093], the author shows how to express the solution to the general Cauchy problem for $a(x, \partial/\partial x)$ in terms of the $U_r^*(\xi, x)$. This result will be used later to study the singularities of the general Cauchy problem.

Finally, the author states that if $a(x, \partial/\partial x)$ is hyperbolic then we can express the elementary solution in terms of U_r^* and quadratures. *L. Ehrenpreis* (Waltham, Mass.)

159:

Hörmander, Lars. Local and global properties of fundamental solutions. Math. Scand. **5** (1957), 27–39.

Soit $P(D)$ un opérateur différentiel à coefficients constants dans R^n. On désigne par L_c^2 (resp. L_{loc}^2) l'espace des fonctions L^2 à support compact (resp. localement L^2). Soit E une solution élémentaire de $P(D)$, i.e., une distribution E telle que $P(D)E=\delta$, masse de Dirac. Malgrange a montré qu'il existe toujours des distributions E solutions élémentaires, telles que pour tout $f\in L_c^2$ on ait $E*f\in L_{loc}^2$. Si $Q(D)$ est un deuxième opérateur différentiel et si l'on suppose que $Q(D)(E*f)\in L_{loc}^2$ pour tout $f\in L_c^2$, alors $Q(D)$ est plus faible que $P(D)$ [pour cette notion, cf. L. Hörmander, Acta Math. **94** (1955), 161–248; MR **17**, 853].

Ceci conduit à la définition et au problème: une solution élémentaire E est dite propre si pour tout $Q(D)$ plus faible que $P(D)$ on a $Q(D)(E*f)\in L_{loc}^2$ pour tout $f\in L_c^2$; existe-t-il des solutions élémentaires propres? L'A. démontre: quel que soit $\varepsilon>0$, il existe E solution élémentaire de $P(D)$ avec les propriétés: a) E est propre; b) $E/\cosh(\varepsilon(x^2+1)^{\frac{1}{2}})$ est tempérée [cf. L. Schwartz, Théorie de distributions, Hermann, Paris, 1950–51; MR **12**, 31, 833]. La démonstration utilise la transformation de Laplace, des estimations sur des fonctions analytiques et le théorème de Hahn–Banach; c'est un raffinement de la méthode de Malgrange.

Si maintenant on suppose que $P(D)$ est hypo-elliptique, deux solutions élémentaires diffèrent d'une fonction indéfiniment différentiable, donc toutes les solutions élémentaires sont propres. L'A. démontre la réciproque.

Si E est solution élémentaire propre de $P(D)$, on a: $Q(D)E\in L_{loc}^2$ si et seulement si $\int \tilde{Q}(\xi)^2/\tilde{P}(\xi)^2 d\xi<\infty$ (où $\tilde{P}(\xi)^2=\sum_\alpha |P^{(\alpha)}(\xi)|^2$, $P^{(\alpha)}(\xi)$=dérivée de $P(\xi)$).

L'A. examine ensuite les relations entre solutions élémentaires propres et tempérées. — L. Schwartz a émis la conjecture que, pour tout $P(D)$, il existe une solution élémentaire tempérée (problème de la division). (Ce problème était ouvert au moment de la rédaction de cet article, mais depuis, L. Hörmander et Łojasiewicz ont, indépendamment, démontré la conjecture de Schwartz.) L'A. démontre qu'il n'existe pas, en général, de solution élémentaire à la fois propre et tempérée.

J. L. Lions (College Park, Md.)

160:

***Bandyopadhyay, G. On certain lemma in connection with separable solutions of partial differential equations.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 269–270. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

In the equation

$$f_1(x)\phi_1(y)\psi_1(z)+f_2(x)\phi_2(y)\psi_2(z)+f_3(x)\phi_3(y)\psi_3(z)=0$$

each function is assumed differentiable and depends only on the variable in its argument. Then it is easily shown that either $f_1=Af_2=Bf_3$, $\phi_1=C\phi_2=D\phi_3$ and $\psi_1+\psi_2/AC+\psi_3/BD=0$ hold, with constants A, B, C and D, or else a similar formula is valid with ψ_i replaced by f_i or ϕ_i. Applications to the method of separation of variables in differential equations are mentioned.

Yu Why Chen (Detroit, Mich.)

161:

Rapoport, I. M. On estimation of eigenvalues of self-adjoint boundary problems. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 3–14. (Ukrainian)

162:

Hong, Imsik. On an eigenvalue and eigenfunction problem of the equation $\Delta u+\lambda u=0$. Kōdai Math. Sem. Rep. **9** (1957), 179–190.

Let C be a smooth simple closed curve in the x, y plane, C^* a closed point set in the interior of C, D the domain bounded by C and C^*. Consider any sequence of domains D_n with the following properties: $D_n\subset D_{n+1}$, $\lim_{n\to\infty} D_n=D$, and the boundary of D_n consists of C and a finite number of smooth curves C_n enclosing C^* in their interior. For each D_n let λ_n be the first eigenvalue, and u_n, with $\iint_{D_n} u_n^2 dxdy=1$, be the first eigenfunction, of the differential equation (1) $u_{xx}+u_{yy}+\lambda u=0$ with $u=0$ on $C+C_n$. The following theorem is proved: Both $\lim_{n\to\infty}\lambda_n=\rho$ and $\lim_{n\to\infty} u_n=v$ exist and are independent of the sequence D_n. Indeed, v is the solution of (1) with $\lambda=\rho$, $\iint_D v^2 dxdy=1$, and with $v=0$ on the boundary of D, except on a set of capacity zero. The latter set is identical with the set of exceptional points of the Green's function for D. The main point of the proof is to show the uniform boundedness of u_n in D and that of the derivatives $\partial u_n/\partial x$ and $\partial u_n/\partial y$ in the interior regions. *Yu Why Chen* (Detroit, Mich.)

163:

Iljin, V. A. Uniform convergence of expansions in characteristic functions for regions having an odd number of dimensions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 650–652. (Russian)

Let g be a domain in an N-dimensional Euclidean space, where N is odd. Expansions of arbitrary functions f with respect to the eigenfunctions of $\Delta u+\lambda u=0$ are considered, the main theorem being as follows. Suppose that g is such as to allow the application of Green's formula to the eigenfunctions, and that the function f, defined in g, satisfies the conditions: (i) f belongs to Sobolev's class $W_p^{[(N-1)/2]}$ for some $p>2N/(N-1)$, (ii) the functions f, Δf, $\cdots$, $\Delta^r f$, $\cdots$ up to the order $r=[(N-2)/4]$ (in the case of the first boundary value problem) or up to the order $r=[(N-4)/4]$ (in the case of the second and third boundary value problems) satisfy the corresponding homogeneous boundary conditions in the generalized sense (i.e., in the mean). Then the expansion of f in terms of the eigenfunctions of $\Delta u+\lambda u=0$ is uniformly convergent in every strictly interior domain g'. The author gives a brief sketch of the proof and indicates that the results are best possible.

F. Smithies (Cambridge, England)

164:

Kopaček, I. On the Dirichlet problem for elliptic equations with a small parameter in the highest derivatives. Uspehi. Mat. Nauk (N.S.) **12** (1957), no. 5(77), 211–220. (Russian)

The Dirichlet problem for

(*) $\varepsilon\sum A_{ij}(x)\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}+\sum A_i(x)\frac{\partial u_\varepsilon}{\partial x_i}+C(x)u_\varepsilon=F(x)$

is considered in a domain G with $u_\varepsilon=\varphi$ on S, the boundary of G. It is assumed that $C(x)\leqq C_0<0$ and $C(x)-\sum \partial A_i/\partial x_i \leqq C_0<0$ in $\bar{G}$. The author proves that as $\varepsilon\to 0$, u_ε converges weakly in $L_2(G)$. Moreover, u is a weak solution of the degenerate equation of the first order obtained by setting $\varepsilon=0$ in (*) and u is subject to an appropriate weak boundary condition on part of the boundary of G. For the first order problem the boundary is divided into three parts depending on whether $\sum A_i n_i$ is >0, <0 or $=0$, where n is the exterior normal, and u satisfies conditions only on the appropriate part.

N. Levinson (Cambridge, Mass.)

165:

***Brousse, Pierre. Résolution de divers problèmes du type Stokes-Beltrami posés par la technique aéronautique.** Préface de H. Poncin. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 323, 1956. ix+67 pp. 1200 francs.

Bien que publié dans la collection du ministère de l'air français, il s'agit d'un mémoire essentiellement mathématique consacré à l'étude de problèmes aux limites de caractère elliptique relatifs à des équations aux dérivées partielles du type Euler-Poisson-Darboux. C'est une œuvre de synthèse dans laquelle l'auteur apporte certaines contributions originales. Un chapitre est consacré aux théorèmes d'existence et d'unicité dans le cas où une partie de la frontière est singulière. Le groupe de la géométrie de Poincaré attaché à l'équation est mis en évidence; il conduit à une étude précise du cas où la frontière est une "droite" de cette géométrie. On trouvera également une étude des représentations intégrales des solutions, des développements sur les séries de Fourier généralisées et l'application de ces résultats à quelques cas particuliers notables. *P. Germain* (Paris)

166:

Dianine, S. A. Sur la résolution du problème de Raman-Dey. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 116–141.

The author calls "problem of Raman-Dey" the finding of solutions of the equation

$$\frac{\partial^2 u}{\partial t^2}-\alpha^2\frac{\partial^2 u}{\partial x^2}=f(x,t)\rho(u),$$

satisfying certain additional conditions.

The equation occurs in the theory of the construction of certain musical instruments. The function ρ is to be determined experimentally. In the case, important for the applications here in view, that $\rho(u)$ is an exponential function, the equation is an extension of Liouville's equation $\partial^2 w/\partial\xi\partial\eta=e^{\kappa w}$ obtained by substituting in the first stage $\xi=x+\alpha t$, $\eta=x-\alpha t$. As is well known, this equation admits the solution

$$e^{\kappa w}=\frac{2A_1'(\xi)A_2'(\eta)}{\kappa\{A_1(\xi)+A_2(\eta)\}^2}.$$

The author considers in the first place Cauchy's problem, to satisfy initial conditions, for $t=0$, $u=F(x)$, $\partial u/\partial t=G(x)$. Then the conditions are established for the functions $F(x)$ and $G(x)$ in order that the solution satisfies given boundary conditions for $x=0$.

Because this method leads generally to great complications, the author also gives solutions satisfying certain boundary conditions for $x=0$ and for $x=l$, without taking into account, in the first instance, given initial conditions. In the last section a number of examples are treated. *H. Bremekamp* (Delft)

167:

Kamenomostskaya, S. L. On a problem of infiltration theory. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 18–20. (Russian)

The author obtains uniqueness and existence proofs for the partial differential equations (found in infiltration theory) $\partial p/\partial t=a_1^2\partial^2 p/\partial x^2$ for $\partial p/\partial t\geqq 0$, and $\partial p/\partial t=a_2^2\partial^2 p/\partial x^2$ for $\partial p/\partial t<0$, subjects to the condition $p(x,0)=p_0(x)$ $(0<x<\infty)$, $\partial p(0,t)/\partial x=0$ $(t>0)$. However, the proofs are based on the properties of the equivalent equation $\alpha\partial p/\partial t+\beta|\partial p/\partial t|=\partial^2 p/\partial x^2$, $\alpha+\beta=1/a_1^2$, $\alpha-\beta=1/a_2^2$, $\alpha>0$, $|\beta|<\alpha$. Here, $p(x,0)=\varphi(x)$ $(-\infty<x<\infty)$; $\varphi(x)$ =even function of x, equal to $p_0(x)$ for $x>0$.

The author's proofs are based on previous results of E. Rothe [Math. Ann. **102** (1930), 650–670], S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565], and O. A. Ladyženskaya [The mixed problem for a hyperbolic equation, Gostehizdat, Moscow, 1953; MR **17**, 160]. *K. Bhagwandin* (Oslo)

168:

Moisil, Gr. C. Configuration caractéristique des systèmes de deux équations aux dérivées partielles linéaires, du deuxième ordre, à trois variables indépendantes. Com. Acad. R. P. Romîne **7** (1957), 689–692. (Romanian. Russian and French summaries)

The author describes the equation induced by the given system in a plane. The proof depends on an earlier paper of the author [# 43 above].

169:

Chen, Y. W. Discontinuity of solutions of quasi-linear differential equations in two variables. Comm. Pure Appl. Math. **9** (1956), 373–381.

The first part of this paper demonstrates in a very simple manner the non-linear behavior of the solution of a problem previously studied by the author [same Comm. **5** (1952), 57–86; **6** (1953), 179–229; MR **14**, 218; **15**, 177]. In the second part are discussed various questions concerning the equation of minimal surfaces. The problem is to find a solution $z(x,y)$ of

$$(1+z_y^2)z_{xx}-2z_xz_yz_{xy}+(1+z_x^2)z_{yy}=0$$

in the exterior of a convex curve C with a finite number of protruding corners, such that it has a preassigned normal direction at infinity and $\partial z/\partial n=0$ on C. For details of some proofs see Chen, Proc. conference on differential equations, Univ. of Maryland, College Park, 1956, pp. 115–138 [MR **19**, 203]. *From the introduction*

170:

Sanders, James. A generalized function-theory and the related Dirichlet problem. I. Ann. of Math. (2) **64** (1956), 523–543.

This paper is concerned with the partial differential equation $N(N(u))=0$, where $N(u)\equiv(\sigma(x))^{-1}(\sigma(x)u_x)_x+\tau(y)((\tau(y))^{-1}u_y)_y$, which includes the biharmonic partial differential equation $\Delta\Delta u=0$ as the special case $\sigma=\tau\equiv 1$. A function theory, bearing roughly the same relation to the equation $N(u)=0$ as the classical theory of analytic functions of one complex variable bears to Laplace's equation $\Delta u=0$, was constructed by L. Bers and A.

Gelbart [Trans. Amer. Math. Soc. **56** (1944), 67–93; MR **6**, 86]. A method for attaching a function theory to the biharmonic equation was given by L. Sobrero [Theorie der ebenen Elastizität..., Teubner, Leipzig-Berlin, 1934], using hypercomplex numbers. This last approach, together with the procedure of Bers and Gelbart, was used by J. B. Diaz [Amer. J. Math. **68** (1946), 611–659; MR **8**, 466] to obtain a function theory associated with the n times iterated equation $N^n(u)=0$. The author proceeds directly to the construction of a function theory for the equation $N^2(u)=0$, as follows. A function $u(x, y)-iv(x, y)$ is said to be Σ_2-monogenic in a domain D of the $z=x+iy$ plane provided that u and v are four times continuously differentiable in D (i.e., $u, v \in C^4$) and satisfy the system of equations $\sigma(x)u_y-\tau(y)v_x=0$, $\sigma(x)u_x+\tau(y)v_y=\theta$, $N'(\theta)=0$, where $\sigma, \tau \in C^3$, $\sigma>0$, $\tau>0$, $\theta \in C^2$, and the operator N' is given by $N'(w)\equiv\sigma(\sigma^{-1}w_x)_x+\tau(\tau^{-1}w_y)_y$ for any function w. For such functions, differentiation and integration processes are defined and are shown to be inverses of each other. By repeated integration of some simple functions, a series of functions are defined which are generalizations of the usual powers $a(z-z_0)^n$, and a generalization of Taylor's theorem is proved. Next, a fundamental solution $\Phi(x, y; \xi, \eta)$ of $N^2(u)=0$ is obtained, and an analogue of Cauchy's integral formula $f(t)=(2\pi i)^{-1}\int_C f(z)(z-t)^{-1}dz$ is derived. The paper concludes with indications as to how the theory can be extended to $N^n(u)=0$ and to certain non-iterated equations. *J. B. Diaz* (College Park, Md.)

171:

Sanders, James. A generalized function-theory and the related Dirichlet problem. II. Ann. of Math. (2) **66** (1957), 141–154.

In the first paper of this series [see the preceding review] the author considered the equation $N^2(u)=0$ and constructed a function theory related to this equation. The present paper is devoted to the solution of the Dirichlet problem for the equation $N'^2(u)=0$. The method used is patterned after the "double layer potential" method of solving the Dirichlet problem for Laplace's equation. G. Lauricella [Acta Math. **32** (1909), 201–256] and K. Schröder [Math. Z. **48** (1943), 553–675; MR **5**, 67] employed a similar idea in dealing with the Dirichlet problem for the biharmonic equation. The author proceeds differently, using the function theory which he developed in his first paper. It is shown that the Dirichlet problem in question is equivalent to the determination of a certain Σ_2-monogenic function from its values on the boundary. The main result concerns the Dirichlet problem which consists of finding a solution u of $N'^2(u)=0$, of class C^4 in a domain D, of class C^1 in the closure of D, which on the boundary C of D satisfies $u=f(s)$, $\partial u/\partial n=g(s)$, where f and g are given continuous functions of the arc length s on C. It is found that this problem possesses one and only one solution, under suitable differentiability assumptions relative to the boundary C and the functions σ and τ.

J. B. Diaz (College Park, Md.)

172:

De Giorgi, Ennio. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.

In this paper, the author proves the following important fundamental result concerning the solutions of uniformly elliptic ($m|\lambda|^2\leq a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta<M|\lambda|^2$; $0<m<M$) equations of the form $\sum \partial(a^{\alpha\beta}\partial u/\partial x^\beta)/\partial x^\alpha)=0$ in ν independent variables where $\nu>2$: Any solution which is in L_2 on a bounded domain G satisfies a uniform Hölder condition on each interior domain D, the exponent in which depends only on ν and M/m and the coefficient in which depends only on these quantities, the L_2 norm of u, and the distance of D from the boundary of G. Actually the author proves this for the case that the coefficients are merely bounded and measurable, in which case the equation must be written in integrated form and the solution u is assumed to be of class $\mathfrak{B}_2$ on interior domains. This result generalizes, to the case where $\nu>2$, an old result of the reviewer for $\nu=2$ [Trans. Amer. Math. Soc. **43** (1938), 126–166] which he used to prove the differentiability of the solutions of certain variational problems and which Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367] used to establish the existence of the solutions of certain quasi-linear elliptic equations. John Nash [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 754–758; MR **19**, 749] has recently proved independently a similar result in connection with his work on parabolic equations. However, his methods are completely different from those of De Giorgi, those of De Giorgi being simpler. Of course De Georgi uses this result to establish the Hölder continuity of the first derivatives of the solutions of certain regular variational problems from which further differentiability follows from known results of E. Hopf [Math. Z. **34** (1931), 194–233], the reviewer [Contributions to the theory of partial differential equations, Princeton Univ. Press, Princeton, pp. 101–159; MR **16**, 827], and Stampacchia [Ricerche Mat. **1** (1952), 200–226; MR **15**, 328].

C. B. Morrey, Jr. (Berkeley, Calif.)

173:

Boboc, Nicolas; et Radu, Nicolas. Sur l'existence de la fonction de Green pour les équations du type elliptique, définies sur des variétés différentiables. C. R. Acad. Sci. Paris **246** (1958), 3204–3207.

174:

Friedman, Avner. Uniqueness properties in the theory of differential operators of elliptic type. J. Math. Mech. **7** (1958), 61–67.

In questo lavoro l'A. porta un interessante contributo al problema ben noto [T. Carleman, C. Müller, E. Heinz, P. Hartman ed A. Wintner, P. Lax, N. Aronszajn, H. Cordes; alle referenze date dall'A. si può ora aggiungere: N. Aronszajn, J. Math. Pures Appl. (9) **36** (1957), 235–249; MR **19**, 1056] e tuttora non risolto, nella sua generalità, dell'unicità della soluzione delle equazioni ellittiche avente uno zero d'ordine infinito.

Sia Lu un operatore differenziale lineare reale di ordine s definito ed ellittico in un dominio limitato B, dello spazio reale euclideo ad n dimensioni, contenente l'origine; con nomenclatura abituale

$$Lu=\sum_{p_1+p_2+\cdots+p_n\leq s} b_p(x)D^p u(x),\ x=(x_1, \cdots, x_n),$$

$$p=(p_1, \cdots, p_n),\ D^p=\partial^{p_1+\cdots+p_n}/\partial x_1^{p_1}\cdots\partial x_n^{p_n},$$

con $\sum_{p_1+\cdots+p_n=s} b_p(x)\xi_1^{p_1}\cdots\xi_n^{p_n}>0$ per ogni vettore reale e non nullo $\xi=(\xi_1, \cdots, \xi_n)$. I coefficienti $b_p(x)$ abbiano derivate continue fino all'ordine $p_1+\cdots+p_n$. Si ha allora il seguente teorema: se $u(x)$ è una funzione continua in B con le derivate fino all'ordine s e tale che I) $Lu(x)\leq 0$, $u(x)\geq 0$ in B, II) per ogni m intero positivo risulti $\sum_{p_1+\cdots+p_n<s}|D^p u(x)|=O(|x|^m)$; allora $u(x)\equiv 0$ in B. La dimostrazione si basa sulla seguentema ggiorazione:

$$\int_{|x|\leq R} u(x)|x|^{-m-s}dx\leq c/mR^{m+s}\ (c \text{ costante}),$$

combinata con un'idea di Carleman. Vengono indicate anche diverse generalizzazioni e contro-esempi. Interessante è l'osservazione che il teorema si può in un certo senso invertire; ad es. si ha che, se l'unica funzione continua con le derivate di ordine $\leqq s$ e verificante le I) e II) è la funzione identicamente nulla, allora si ha

$$\sum_{p_1+\cdots+p_n=s} b_p(0)\xi_1^{p_1}\cdots\xi_n^{p_n}\geqq 0$$

per ogni vettore reale ξ. *E. Magenes* (Genova)

175:

Roždestvenskiĭ, B. L. On systems of quasilinear equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 454–457. (Russian)

The author considers the system of quasilinear equations

$$(1)\qquad \frac{\partial u_i}{\partial t}+\sum_{j=1}^{n} a_{ij}\frac{\partial u_j}{\partial x}=b_i\ (i=1,2,\cdots,n),$$

where a_{ij}, b_i are arbitrary analytic functions of the u_j, t, x. This system is assumed to be hyperbolic in the domain of u_j, t, x under consideration. It is assumed that the system (1) may be reduced to the form

$$(2)\ \frac{\partial \phi_i}{\partial t}(u_j,t,x)+\frac{\partial \psi_i}{\partial x}(u_j,t,x)=f(u_j,t,x)\ (i=1,2,\cdots,n),$$

The generalized solution of the problem of Cauchy for system (1) is the set of functions $u_j(t,x)$ taking on the given values on an initial curve and satisfying the integral relations

$$(3)\ \int_C \phi_i(u_j,t,x)dx-\psi_i(u_j,t,x)dt=\iint_G f_i(u_j,t,x)dtdx,$$

where C is an arbitrary contour bounding the domain G in (x,t) space.

If the equation

$$(4)\qquad \frac{\partial \phi}{\partial t}(u_j,t,x)+\frac{\partial \psi}{\partial x}(u_j,t,x)=f(u_j,t,x)$$

is a consequence of system (1), then it is called the law of conservation for system (1). If (1) has not less than n independent laws of conservation it is called conservative; otherwise, non-conservative. If there are precisely n conservative equations it is called conservative-closed and if system (1) does not have a single conservative equation it is said to be completely non-conservative.

Then the following statements hold. (I) System (1) for $n\leqq 2$ is always conservative. (II) System (1) for $n\geqq 3$, as a rule, is completely non-conservative. (III) The property of the system of quasilinear equations (1) being conservative is preserved by any non-degenerate transformation depending on k independent variables.

C. G. Maple (Ames, Iowa)

176:

Lax, P. D. Hyperbolic systems of conservation laws. II. Comm. Pure Appl. Math. **10** (1957), 537–566.

The author investigates systems of the form

$$(*)\qquad u_t+Au_x=0,$$

where $u(x,0)=\phi(x)$; here x, t are two independent variables, $u(x,t)$, $\phi(x)$ are vectors, and $A(u)$ is a matrix. Usually $A=\operatorname{grad} f(u)$, and (*) is a "conservation law". Continuing earlier work [same Comm. **7** (1954), 159–193; MR **16**, 524], the author discusses "weak solutions" of (*) and unique solutions of the initial-value problem. For a single conservation law, he gives explicitly a weak solution of the initial-value problem. He discusses the asymptotic behavior of solutions as t tends to infinity. Turning to systems of conservation laws (the time-dependent supersonic flow equations are an example), the author generalizes the entropy condition and develops a theory of shocks. He gives a theory of simple waves for (*). The paper concludes with a solution of "Riemann's problem" (initial function piece-wise constant). There is an extensive bibliography.

M. A. Hyman (Yorktown Heights, N.Y.)

DIFFERENTIAL ALGEBRA

177:

***Kaplansky, Irving. An introduction to differential algebra.** Actualités Sci. Ind., No. 1251=Publ. Inst. Math. Univ. Nancago, No. 5. Hermann, Paris, 1957. 63 pp. 800 francs.

This little book grew out of an introductory course on differential algebra given by the author. It is, appropriately, a lucid exposition of limited portions of the subject. The portions selected by him are, in the first place, the Picard-Vessiot theory (serving as an introduction to the Galois theory of differential fields), and, secondly and more briefly, the basis theorem and its more immediate consequences (serving as an introduction to Ritt's general theory of algebraic differential equations). For the sake of simplicity, and of economy in time and space, consideration is limited to those differential fields which are ordinary and of characteristic zero. The presentation is concise and easy to read. The facts about algebraic matric groups needed in the Picard-Vessiot theory are developed almost entirely in an original abstract form in the framework of general topology; these considerations are carried forward, in an appendix, beyond the point needed in the rest of the text.

E. R. Kolchin (New York, N.Y.)

178:

Seidenberg, A. Abstract differential algebra and the analytic case. Proc. Amer. Math. Soc. **9** (1958), 159–164.

The author proves a "Lefschetz principle" for differential fields, partial as well as ordinary. Namely, every finitely generated differential field extension of the differential field of rational numbers is isomorphic to a differential field of meromorphic functions. This permits the extension to arbitrary differential fields of characteristic zero theorems which are proved in the analytic case and which are of "finite type".

E. R. Kolchin (New York, N.Y.)

POTENTIAL THEORY

See also 120, 170, 171, 207, 229, 440.

179:

Obreschkoff, Nikola Über harmonische Funktionen in einem Riemannschen Raum. Schr. Forschungsinst. Math. **1** (1957), 264–268.

Questo lavoro è un compendio di tre precedenti lavori dell'autore [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4** (1943), 133–136, 359–362, 420–423; Ann. Univ. Sofia. Fac. Phys.- Math. Livre 1. **39** (1943), 133–216; **40** (1944), 131–137; MR **8**, 271; **11**, 176]. *L. Gori* (Roma)

180:

Laurenti, Fernando. Funzioni armoniche in un cerchio e che sul contorno coincidono con una funzione razionale data. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 40–55.

Viene calcolata direttamente senza ricorrere all'integrale di Poisson la funzione armonica in un cerchio del piano (x, y), che sulla circonferenza frontiera coincide con una assegnata funzione razionale di x e y.

E. Magenes (Genova)

181:

Afanas'ev, N. L.; and Kučin, V. P. Determination of the position and dimensions of a charged ellipsoid with three unequal axes and of an ellipsoid of revolution about a horizontal axis from measurements of the gradient of its potential. Izv. Dnepropetr. Gorn. Inst. **23** (1955), 65–70. (Russian)

From known expressions for the gradient of the potential of a charged ellipsoid a method is derived for determining its center and semi-axes from measurements of the gradient on a surface parallel to its axis and passing over it.

V. I. Levin (RŽMat **1956**, # 3902)

182:

Riiber, Ågot E. Über die Kapazität einer verallgemeinerten Cantorschen Punktmenge. Arch. Math. **8** (1957), 368–373.

Let M_0 be a non-degenerate bounded closed connected plane set, and $M_1 \subset M_0$ be the union of k_1 non-degenerate closed connected subsets $A_{1,\nu}$ $(\nu=1, 2, \cdots, k_1)$ which are disjoint from each other. We continue the process and M_n consists of $k_1k_2\cdots k_n$ non-degenerate closed connected sets $A_{n,\nu}$ such that each $A_{n-1,\nu}$ contains exactly k_n sets $A_{n,\nu}$ which are disjoint from each other. We call two $A_{n,\nu}$ associated with each other when both belong to the same $A_{n-1,\nu}(A_0=M_0)$. We assume that the maximum d_n' of the diameters $D_{n,\nu}$ of $A_{n,\nu}$ decreases to 0 as $n\to\infty$ $(D_0=d_0'=d_0)$. The set $M=\bigcap_{n=0}^{\infty} M_n$ is then called a generalized Cantor set. The author gives criteria for M to be of capacity zero; the capacity in this paper means the logarithmic capacity. She assumes throughout the paper that $\sup_n d_n'/d_{n-1}'<1$. We set $1/p_n=k_nd_n'/d_{n-1}'$ (this definition is slightly different from hers and p_n can be ≤ 1), $P_n=p_1p_2\cdots p_n$ $(P_0=1)$ and $N_n=k_1k_2\cdots k_n$ $(N_0=1)$. Th. 1: If $\sum_\nu N_\nu^{-1}\log p_\nu=\infty$ then the capacity of M is zero. Next we set $r_{n+1}=\delta_{n+1}/d_n'<1$ where δ_{n+1} is the smallest distance between two $A_{n+1,\lambda}$ associated with each other. Th. 2: Suppose that $-\sum_{\nu=0}^{\infty} N_\nu^{-1}\log r_{\nu+1}<\infty$. Then the capacity of M is positive if and only if $\sum_{\nu=1}^{\infty} N_\nu^{-1}\log p_\nu<\infty$. In the last theorem it is proved that the capacity of M of special type related to the general Cantor set discussed by the reviewer [Ann. Inst. Fourier, Grenoble **5** (1953–54), 1–28; Nagoya Math. J. **11** (1957), 151–160; MR **17**, 1191; **19**, 541] and G. af Hällström [Acta Acad. Abo. **20**, no. 5 (1955); MR **17**, 146] is positive.

M. Ohtsuka (Hiroshima)

183:

Bugrov, Ya. S. Dirichlet problem for the circle. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 639–642. (Russian)

The polyharmonic Dirichlet problem (of order l) for the unit circle consists of finding a function U in C^{l-1} on the closed disc, satisfying

$$\Delta^l U(\rho, \vartheta)=0 \quad (0\leq\rho<1;\ 0\leq\vartheta\leq 2\pi),$$

$$\left.\frac{\partial^k U(\rho, \vartheta)}{\partial\rho^k}\right|_{\rho=1}=\varphi_k(\vartheta) \quad (k=0, 1, \cdots, l-1),$$

where Δ is the Laplace operator and $\varphi_0, \varphi_1, \cdots, \varphi_{l-1}$ are preassigned boundary functions. In this note it is shown that if φ_k belongs to the Nikol'ski class $H_p^{r+l-(k+1)}(M)$ $(k=0, 1, \cdots, l-1)$ for $r>0$ (non-integral if $l>1$) and $1\leq p\leq\infty$, then the polyharmonic function U belongs to $H_p^{r+1/p+l-1}(cM)$ for some positive constant c.

M. G. Arsove (Seattle, Wash.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 188, 768.

184:

Fort, Tomlinson. Linear difference and differential equations satisfying conditions at more than one point. Proc. Amer. Math. Soc. **9** (1958), 287–292.

A system consisting of a finite number of linear, homogeneous, first-order difference equations is considered for cases where assigned conditions on the solution components may occur at more than one point. Sufficient hypotheses are given for the existence of solutions. Through passage to the limit from the algebraic system to a related differential system, similar results are obtained for the linear, homogeneous, differential system with conditions at more than one point. The non-vanishing of certain determinants formed from the coefficients are the essential hypotheses needed for the theorems. R. Conti and others found similar conditions to be needed for their existence theorems for certain non-linear systems with assigned values at more than one point [see reference cited in the paper under review].

W. M. Whyburn (Chapel Hill, N.C.)

185:

Anastassiadis, Jean. Sur les solutions logarithmiquement convexes ou concaves d'une équation fonctionnelle. Bull. Sci. Math. (2) **81** (1957), 78–87.

Verf. verallgemeinert ein bekanntes Resultat bezüglich der Γ-Funktion [E. Artin, Einführung in die Theorie der Gammafunktion, Teubner, Leipzig-Berlin, 1931]; in der vorliegenden Arbeit wird auch ein Variant dieses Beweises auf Grund eines Satzes von P. Montel [J. Math. Pures Appl. (9) **7** (1928), 29–60] gegeben. Er beweist nämlich, dass die einzige Lösung der Differenzengleichung

$$f(x+1)=\frac{x(x+a_1)(x+a_2)\cdots(x+a_p)}{(x+b_1)(x+b_2)\cdots(x+b_q)}f(x)$$
$$(a_i>0,\ b_j>0;\ i=1, 2, \cdots, p;\ j=1, 2, \cdots, q),$$

die für genügend grosse x logarithmisch konvex [konkav] bleibt, für $x>0$

$$f(x)=\frac{\Gamma(x)\Gamma(x+a_1)\Gamma(x+a_2)\cdots\Gamma(x+a_p)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)a_1a_2\cdots a_p}$$
$$\times\frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)b_1b_2\cdots b_q}{\Gamma(x+b_1)\Gamma(x+b_2)\cdots\Gamma(x+b_q)}$$

ist. Verf. stellt noch die Bedingung $p>q-1$ oder $p=q-1$ und $b_1+b_2+\cdots+b_q>a_1+a_2+\cdots+a_p$ [bzw. $p<q-1$ oder $p=q-1$ und $b_1+b_2+\cdots+b_q<a_1+a_2+\cdots+a_p$], diese braucht er aber nur um von einer Ungleichungskette, die er schon bewiesen hat, noch zu zeigen, dass die Ungleichung zwischen dem ersten und dem letzten Glied tatsächlich besteht. Diese zusätzliche Bedingung scheint vielmehr dafür notwendig zu sein, dass es überhaupt für genügend grosse x logarithmisch konvexe [konkave] Lö-

sungen gebe. Es wird auch der Fall, wo die Voraussetzungen $a_i>0$, $b_j>0$, $x>0$ nicht mehr erfüllt sind, untersucht. Hier ist die bezügliche zusätzliche Bedingung schon wesentlicher, da der Beweis der oben erwähnten Ungleichungskette für $x<0$ nicht mehr richtig ist. Der Beweis ist übrigens klar geführt und leicht verständlich. Als Spezialfälle figurieren die Eulersche Betafunktion und gewisse speziale hypergeometrische Funktionen. {Eine Anzahl von Druckfehlern stört das Lesen. Ref. bemerkt, dass ähnliche Fragen von einem sehr allgemeinen Gesichtspunkt aus schon untersucht wurden [W. Krull, Math. Nachr. 1 (1948), 365–376; 2 (1949), 251–262; MR 11, 112, 364].} *J. Aczél* (Debrecen)

186:

Anastassiadis, Jean. Une propriété de la fonction Gamma. Bull. Sci. Math. (2) 81 (1957), 116–118.

Es wird bewiesen, dass die einzige Lösung der Differenzengleichung $f(x+1)=xf(x)$ unter der Anfangsbedingung $f(1)=1$, für die $(e/x)^x f(x)$ bei genügend grossen x abnimmt, die Gammafunktion $f(x)=\Gamma(x)$ ist.

{Ein Druckfehler: auf S. 118, Zeilen 9–10 von unten ist $[(n+x)/(n+x+1)]^{n+x+1}$ kleiner und nicht grösser als 1.} *J. Aczél* (Debrecen)

187:

Ghermanescu, Michel. Caractérisation fonctionnelle des fonctions trigonométriques. Bul. Inst. Politech. Iaşi 4 (1949), 362–368.

The author studies functional equations suggested by the addition theorems for the functions sine and cosine. Assuming only measurability of the solutions he arrives routinely at the conventional conclusions.

J. G. Wendel (Ann Arbor, Mich.)

SEQUENCES, SERIES, SUMMABILITY

See also 21, 280.

188:

Azpeitia, A. G. Interpolation functions of the coefficients of a recurring series. Rev. Acad. Ci. Madrid 51 (1957), 11–19. (Spanish. English summary)

This expository article provides a rigorous treatment of recurring series and their generating functions, making use of the theory of linear difference equations with constant coefficients. *T. N. E. Greville* (Washington, D.C.)

189:

Maravić, M. Sur un procédé de sommation des séries divergentes. Srpska Akad. Nauka. Zb. Rad. 55, Mat. Inst. 6 (1957), 5–52. (Serbo-Croation. French summary)

Le procédé de sommation défini par

$$G_\theta^\kappa(\mu, x)=\sum_{\mu_\nu\leq x}(1-\exp(\mu_\nu-x)x^{-\theta})^\kappa\alpha_\nu,\quad \mu_n\nearrow\infty,\ n\to\infty$$

$(0<\theta<1, \kappa>0)$, dont la portée aux problèmes de sommation de séries de Fourier généralisées a été signifie par V. Avakumović [Math. Scand. 4 (1956), 161–173; MR 18, 315] est étudié dans cette note en detail. I. Les problèmes de type abélienne (théorèmes directs). II. La relation avec la moyenne de Riesz $\sigma_k(x)$, c.-à.-d. G_θ^κ tend vers la même valeur limite pour tous les $\kappa>k$, k un entier. III. La comparaison des procédés G_θ^κ entre eux, pour les différentes valeurs de κ. IV. Théorèmes inverses de sommabilités relatifs au procédé G_θ^κ. V. Applications aux problèmes de sommation des séries de Fourier généralisées.

M. Tomić (Belgrade)

190:

Tull, J. P. The multiplication problem for Dirichlet series. Proc. Amer. Math. Soc. 9 (1958), 332–334.

The paper is concerned with a theorem of E. Landau to the effect that if α, β, λ, μ are real numbers with $\min(\lambda, \mu)>\max(\alpha, \beta)$; and if the series $\sum a_n\xi_n^{-s}$, $\sum b_n\xi_n^{-s}$ converge, respectively, for $\sigma>\alpha$ and $\sigma>\beta$; and the series converge absolutely for $\sigma>\lambda$ and $\sigma>\mu$, respectively; then the Dirichlet product of these two series converges for $\sigma>(\lambda\mu-\alpha\beta)/(\lambda+\mu-\alpha-\beta)$ $(\sigma=\mathrm{Re}\ s)$. The author gives examples of two Dirichlet series, for each α, β, λ, μ with $\min(\lambda, \mu)>\max(\alpha, \beta)$, whose product has abscissa of convergence exactly equal to $(\lambda\mu-\alpha\beta)/(\lambda+\mu-\alpha-\beta)$, thus showing that Landau's theorem is the best possible.

D. Moskovitz (Pittsburgh, Pa.)

191:

Lorentz, G. G.; und Zeller, K. Über Paare von Limitierungsverfahren. Math. Z. 68 (1958), 428–438.

Es seien A und B zwei permanente und für beschränkte Folgen verträgliche Matrizen mit der Eigenschaft: Zu jedem $\varepsilon>0$ gibt es beschränkte Folgen $x=\{x_k\}$ und $y=\{y_k\}$ so dass $A\text{-}\lim x$ bzw. $B\text{-}\lim y$ existiert und $|A\text{-}\lim x-B\text{-}\lim y|>1$ ist, während $\sup_k |x_k-y_k|<\varepsilon$ ist. Diese Eigenschaft bezeichnen die Verf. als Singularität S_1. Es wird gezeigt, dass Matrizen A und B mit der Singularität S_1 existieren, ferner zieht S_1 die Unverträglichkeit von A und B nach sich. Nach Petersen [J. London Math. Soc. 32 (1957), 62–65; MR 18, 733] gibt es für beschränkte Folgen verträgliche und permanente Matrizen A und B, so dass kein Verfahren C stärker als A und B ist. Die Verf. zeigen, dass aber stets ein C existiert, das alle Folgen $z=x+y$ (x bzw. y A- bzw. B-limitierbar und beschränkt) limitiert (und für beschränkte Folgen mit A und B verträglich ist). Es folgen weitere Ergebnisse über starke Limitierbarkeit und Kern-Verträglichkeit.

A. Peyerimhoff (Giessen)

192:

Rajagopal, C. T. On a theorem of Frobenius and Knopp for Abel summability. Math. Z. 67 (1957), 310–319.

For each $r\geq 0$, let $\sigma_n^{(r)}$ denote either the Cesàro or Hölder transform of order r of a real sequence $s_0, s_1, \cdots$. If $0\leq r_0\leq r$, if $\sigma_n^{(r_0+1)}\to L$ as $n\to\infty$, and if $f(t)$ has bounded variation over $0\leq t\leq 1$, then the series in

$$g(y)=y\sum_{n=0}^{\infty} e^{-ny}f(e^{-ny})\sigma_n^{(r)}$$

converges when $y>0$ and

$$g(y)\to L\int_0^1 f(t)dt$$

as $y\to 0+$. To this theorem on transforms of sequences, there corresponds an analogous theorem on transforms of functions $s(u)$ which are Lebesgue integrable over each finite interval $0\leq u\leq u_0$. For each $r\geq 0$, let $\sigma^{(r)}(u)$ denote either the Cesàro or Hölder integral transform of order r of $s(u)$. If $0\leq r_0\leq r$, if $\sigma^{(r_0+1)}(u)\to L$ as $u\to\infty$, and if $f(t)$ has bounded variation over $0\leq t\leq 1$, then the integral in

$$g(y)=y\int_0^\infty e^{-uy}f(e^{-uy})\sigma^{(r)}(u)du$$

exists, as a Cauchy extension of Lebesgue integrals, when $y>0$ and, moreover,

$$g(y)\to L\int_0^1 f(t)dt$$

as $y\to 0+$. For Cesàro transforms, there are similar theorems in which it is supposed that the sequence s_n or function $s(u)$ is evaluable (C,∞) and $f(t)$ is Riemann integrable over $0\leqq t\leqq 1$. *R. P. Agnew* (Ithaca, N.Y.)

193:

Tenenbaum, Morris. Transforms of Tauberian series by Riesz methods of different orders. Duke Math. J. **25** (1957), 181–191.

The Riesz transform $R_1^{(r)}$, $R_2^{(r)}$, $\cdots$ of order r of a series $u_1+u_2+\cdots$ has elements defined by $R_n^{(r)}=\sum_{k=1}^{n}(1-k/n)^r u_k$. Let p and q be real numbers, not necessarily integers, for which $0\leqq p\leqq q$. For each $\alpha>0$, let $m(\alpha)$ and $n(\alpha)$ be positive integers and let $m(\alpha)\to\infty$ and $n(\alpha)\to\infty$ as $\alpha\to\infty$. The first problem is to determine the least constant B for which

$$\limsup_{\alpha\to\infty}|R_n^{(q)}-R_m^{(p)}|\leqq B\limsup_{n\to\infty}n|u_n|$$

whenever $\sum u_n$ is a series for which the right member is finite. The main problem is to study the manner in which B depends upon the constants p and q and the functions $m(\alpha)$ and $n(\alpha)$. In particular, assuming that p and q are fixed, the author characterizes those functions $m(\alpha)$ and $n(\alpha)$ for which B assumes its minimum value. The results include formulas and numerical tables that are too lengthy for a short review. For reasons which the author sets forth, the results provide solutions of problems involving the Cesàro and Abel transforms of series $\sum u_n$ satisfying the Tauberian condition $nu_n=O(1)$.

R. P. Agnew (Ithaca, N.Y.)

194:

Karadžić, Lazar. Quelques conséquences d'un théorème des séries fonctionnelles. Bull. Soc. Math. Phys. Serbie **8** (1956), 29–38. (Serbo-Croatian summary)

Let (α_n) and (β_n) $(n=0, 1, 2, \cdots)$ be two sequences of functions defined on a domain D in the complex plane, and define

$$F(t)=\sum_{n\geqq 0}\beta_n(x)t^n,\quad G(t)=\sum_{n\geqq 0}|\beta_n(x)|t^n,$$

$$\Delta^n\alpha_0(x)=\alpha_n(x)-\binom{n}{1}\alpha_{n-1}(x)+\cdots\pm\binom{n}{n-1}\alpha_1(x)\mp\alpha_0(x).$$

Theorem: If (*) $G^{(i)}(1)=O(1)$ $(i\to\infty)$ and $\Delta^n\alpha_0(x)=o(1)$ $(n\to\infty)$, both uniformly for $x\in D$, then

$$\sum_{n\geqq 0}\alpha_n(x)\beta_n(x)=\sum_{n\geqq 0}\Delta^n\alpha_0(x)F^{(n)}(1)\quad(x\in D).$$

If (*) is false, but instead $G^{(i)}(1)/i!=O(1)$ $(i\to\infty)$, the other conditions remaining as before, then

$$\sum_{n\geqq 0}\alpha_n(x)\beta_n(x)=\sum_{n\geqq 0}\Delta^n\alpha_0(x)F^{(n)}(1)/n!\quad(x\in D).$$

The proof uses iterated partial summations and a basic theorem of Toeplitz. Applications are made to entire functions and their Weierstrassian product representations, and to the study of the behaviour of a power series at points of its circle of convergence — in particular the existence of arcs of the latter on which the series converges uniformly. *R. E. Edwards* (Woking)

APPROXIMATIONS AND EXPANSIONS

See also 134, 411.

195:

Spitzbart, Abraham; and Macon, Nathaniel. On polynomial approximation with deviations in prescribed ratios. Proc. Amer. Math. Soc. **9** (1958), 243–249.

Given the distinct abscissae $x_0, \cdots, x_n$, and the numbers y_i and λ_i, it is required to find a polynomial $y(x)$ of degree $n-1$ such that $y(x_i)=y_i-\lambda_i d$, with d unspecified. The solution is obtained in terms of a matrix W which is nonsingular unless the points (x_i, λ_i) lie on a polynomial of degree $n-1$ or less. Special consideration is given to the case of equally spaced points x_i, and to the case of Chebyshev points.

A. S. Householder (Oak Ridge, Tenn.)

196:

Kiš, O. Remark on mechanical quadrature. Acta Math. Acad. Sci. Hungar. **8** (1957), 473–476. (Russian)

Turan [Acta Sci. Math. Szeged **12** (1950), 30–37; MR **12**, 164] proved that there exist real numbers $\mu_\lambda^{(k)}$ such that the formula

$$\int_0^\pi f(t)dt=\sum_{\lambda=0}^{n}\sum_{k=0}^{2r}\mu_\lambda^{(k)}f^{(k)}\left(\frac{2\lambda-1}{2n}\pi\right)$$

holds for all even trigonometric polynomials of order $2(r+1)n-1$, but did not give any expression for the $\mu_\lambda^{(k)}$. The author proves the following formula:

$$\int_0^\pi f(t)dt=\frac{\pi}{n\cdot r!^2}\sum_{\rho=0}^{r}\frac{S_\rho}{4\rho n^{2\rho}}\sum_{\lambda=1}^{n}f^{(2\rho)}\left(\frac{2\lambda-1}{2n}\pi\right),$$

where S_ρ is the $(r-\rho)$th elementary symmetric polynomial of the arguments $1, 4, \cdots, r^2$.

P. Rabinowitz (Rehovoth)

197:

Karadžić, Lazar. Approximation des fonctions par une suite de polynômes. Bull. Soc. Math. Phys. Serbie **8** (1956), 121–124. (Serbo-Croatian summary)

The author uses a result established in an earlier note [≠194 above] to yield the construction of polynomial approximations of functions f holomorphic in a sufficiently large disk with centre at the origin. If the Taylor expansion is $f(z)=\sum_{n\geqq 0}c_n z^n$, it is shown that under suitable conditions the polynomials

$$\sum_{0\leqq n\leqq N}n!c_n\binom{N}{n}(z/n)^n$$

converge to $f(z)$ as $N\to\infty$ for z in the disk of convergence. The process is illustrated by application to the exponential function. There are a number of minor misprints.

R. E. Edwards (Woking)

198:

Natanson, G. I. On the Lozinskiĭ theorem. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 32–35. (Russian)

S. M. Lozinskiĭ [Mat. Sb. N.S. **14(56)** (1944), 175–268; MR **6**, 264] zeigte, daß Matrixverfahren, die die Fourierreihe jeder stetigen Funktion gleichmäßig summieren, eine entsprechende Eigenschaft bezüglich bestimmten Folgen trigonometrischer Interpolationspolynome aufweisen. Verfasser überträgt das auf ultrasphärische Polynome. — Sei $-\frac{1}{2}\leqq\alpha\leqq\frac{1}{2}$ und $J_n(x)=J_n^{(\alpha)}(x)$ das System der orthonormierten Polynome auf $[-1, 1]$ bezüglich $p(x)=(1-x^2)^\alpha$. Einem $f\in C([-1, 1])$ wird einmal die Fourierreihe $\sum a_m J_m(x)$ zugeordnet, zum andern die Folge der Interpolationspolynome $L_n[f; x]=\sum_{m=0}^{n-1}a_m^{(n)}J_m(x)$, die mit f in den Wurzeln $x_k^{(n)}$ von $J_n(x)$ übereinstimmen. Mit Hilfe einer Matrix $\rho_m^{(n)}$ bildet man

$$S_n^{(\rho)}[f;x]=\sum_{m=0}^{n}\rho_m^{(n)}a_mJ_m(x),$$

$$L_n^{(\rho)}[f;x]=\sum_{m=0}^{n-1}\rho_m^{(n)}a_m^{(n)}J_m(x).$$

Verfasser beweist nun den Satz: Ist die Matrix $\rho_m^{(n)}$ so beschaffen, daß $S_n^{(\rho)}[f;x]$ in $[-1+h, 1-h]$ (wo $0\leqq h<1$) gleichmäßig gegen $f(x)$ konvergiert bei beliebigem $f\in C([-1,1])$, so konvergiert für diese f auch $L_n^{(\rho)}[f;x]$ gleichmäßig gegen $f(x)$ in demselben Intervall. — Beim Beweis geht es darum, aus der Beschränktheit der Normen der Operatoren $S_n^{(\rho)}$ die entsprechende Eigenschaft bei den $L_n^{(\rho)}$ zu erschließen, wobei recht scharfe Aussagen über die $x_k^{(n)}$ herangezogen werden.

K. Zeller (Tübingen)

199:

Alexits, Georges. Sur la convergence et la sommabilité des séries orthogonales lacunaires. Acta Sci. Math. Szeged **18** (1957), 179–188.

Given a non-decreasing function $\lambda(x)$, which is concave from below, positive for $x\geqq 1$, and such that $\lambda(x)\leqq x$, the orthogonal series $\sum_0^\infty c_n\varphi_n(x)$ is said to be $\lambda(n)$-lacunary if the number of non-null coefficients in the series with index between n and $2n$ is not greater than $\lambda(n)$. The author gives sufficient conditions for the convergence almost everywhere, or summability (C,α), $\alpha>0$, almost everywhere, of such series. These conditions bear on the order of the coefficients c_n alone, or on c_n and $\lambda(n)$.

K. Chandrasekharan (Bombay)

200:

Natanson, G. I. On the theory of approximation of functions by linear combinations of Sturm-Liouville characteristic functions. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 263–266. (Russian)

Six theorems are stated without proofs which deal mainly with the degree of convergence of the best approximation of a function on an interval $[0,\pi]$ by combinations of eigenfunctions of the Sturm-Liouville problem:

$$U''(x)+[\lambda-B(x)]U(x)=0,$$

$$U'(0)-hU(0)=0,\ U'(\pi)+HU(\pi)=0,$$

where $B(x)$ is given and at least continuous on $[0,\pi]$, h and H are given real numbers. The first theorem says that if $f(x)$ is continuous on $[0,\pi]$,

$$E_n^{SL}(f)=\inf_{c_k}\max_{x\in[0,\pi]}|f(x)-\sum_{k=0}^{n}c_kU_k(x)|=O(1)\Big[\omega\Big(f,\frac{1}{n}\Big)+\frac{1}{n}\|f\|\Big],$$

where U_k are eigenfunctions of the Sturm-Liouville problem, $\omega(f,\delta)=\sup_{|x-y|\leq\delta}|f(x)-f(y)|$ is the modulus of continuity, $\|f\|=\max_{x\in[0,\pi]}f(x)$ and $O(1)$ can be estimated independently from n or $f(x)$.

Theorem 2 gives an estimate of $E_n^{SL}(f)$ in terms of derivatives of $f(x)$. Theorems 3 and 4 represent analogues to theorems given for trigonometric functions by Lozinskiĭ [same Dokl. (N.S.) **83** (1952), 789–792; MR **13**, 838]. Theorem 5 states the conditions under which $E_n^{SL}(f)<A/n$ (A constant independent of n). Finally Theorem 6 contains the estimate for the Lagrange interpolation formula

$$L_n^{SL}[f;x]=\sum_{k=1}^{n}\frac{U_n(x)}{(x-x_k)U_n'(x)}f(x_k),$$

where x_k are roots of $U_n(x)$:

$$f(x)-L_n^{SL}[f;x]=O(\ln n)\Big[\omega\Big(f,\frac{1}{n}\Big)+\frac{1}{n}\|f\|\Big]$$

uniformly on $[a,\pi-a]$ for any $a\in(0,\pi/2)$.

U. W. Hochstrasser (Lawrence, Kansas)

201:

Džrbašyan, M. M.; and Tavadyan, A. B. On weighted-uniform approximation by polynomials of functions of several variables. Mat. Sb. N.S. **43(85)** (1957), 227–256. (Russian)

The authors prove theorems about weighted-uniform approximation of functions $f(x_1,\cdots,x_n)$ by polynomials, which are known for $n=1$, see Izumi and Kawata [Tôhoku Math. J. **43** (1937), 267–273], Šaginyan [Akad. Nauk Armyan SSR. Soobšč. Inst. Mat. Meh. **1** (1947), 3–59], Džrbašyan [Mat. Sb. N.S. **36(78)** (1955), 353–440; MR **17**, 31]. Induction in n and an integral representation of the Steklov averages of f are used. Among the theorems about the degree of approximation we quote the following. If $p(x)$ is an even function of x on $(-\infty,+\infty)$ such that $p(x)=p(1)+\int_1^x t^{-1}\omega(t)dt$ for $x\geqq 1$ with an increasing $\omega(t)$ and $\omega(t)\to\infty$ for $t\to\infty$, we say that $p\in A$. Assume that f and $\partial^{r}f/\partial x_k^{r}$ ($r=1,\cdots,r_k$; $k=1,\cdots,n$) exist and are continuous in the whole space and that $|\partial^{r_k}f/\partial x_k^{r_k}|\leqq M_k$ ($k=1,\cdots,n$) almost everywhere. Let $p_k(x)\geqq 0$ ($k=1,\cdots,n$) satisfy the conditions $p_k\in A$, $x^{-1}p_k(x)\uparrow+\infty$ for $x\to\infty$, and let $x=q_k(y)$ be the inverse of $y=p_k(x)$. Then the degree of approximation of f,

$$E_{m_1,\cdots,m_n}(f;p_1,\cdots,p_n)=\inf\Big\{\max_{x_1,\cdots,x_n}\exp\Big(-\sum_1^n p_k(x_k)\Big)|f(x_1,\cdots,x_n)-P(x_1,\cdots,x_n)|\Big\},$$

where the infimum is taken for all polynomials of degree m_k in x_k, $k=1,\cdots,n$, satisfies

$$E_{m_1,\cdots,m_n}\leqq C\sum_{k=1}^{n}M_k\Big(\frac{q_k(m_k)}{m_k}\Big)^{r_k}.$$

Here an approximation of f by entire functions is used for the proof.

G. G. Lorentz (Syracuse, N.Y.)

FOURIER ANALYSIS

See also 224, 225, 329, 330, 331.

202:

Davidovskiĭ, N. A. On estimation of the approximation of differentiable functions by Bernstein-Rogosinski polynomials. Beloruss. Politehn. Inst. Sb. Nauč. Trudov **44(6)** (1954), 484–486. (Russian)

Let $B_n(f;x)$ be Bernstein-Rogosinki sums of the function f

$$B_n(f;x)=\tfrac{1}{2}\Big[S_n\Big(x+\frac{\pi}{2n+1}\Big)+S_n\Big(x-\frac{\pi}{2n+1}\Big)\Big],$$

where $S_n(x)$ is the Fourier sum of nth order of the function f at the point x. Consider the magnitude

$$E_n(x)=\sup|f(x)-B_n(f;x)|,$$

where the upper limit refers to the class $KW^{(r)}$ of 2π-periodic functions possessing an rth derivative with $|f^{(r)}(x)|\leqq K$. The following formulas are stated without proof:

$$E_n(x)=\frac{2K\pi}{(2n+1)^2}\sum_{k=0}^{\infty}(2k+1)^{-r+1}+O(n^{-3})$$

$$=\frac{2K}{\pi}\int_0^{2\pi}\Big|\sum_{i=1}^{n}\frac{\sin^2 i\pi}{4n+2}\frac{\sin ix}{i^r}\Big|dx+O(n^{-r+1})\qquad(r=5,7,9,\cdots),$$

$$E_n(x)\leqq 2K\pi^2(2n+1)^{-2}+O(n^{-4})\qquad(r=4,6,8,\cdots).$$

In the article there are some inaccurate expressions and omissions. The class $KW^{(r)}$ is not defined with complete accuracy; the statement is made that there exist continuous functions which everywhere fail to have a Fourier expansion, and so forth.

I. G. Sokolov (RŽMat **1956**, 1235)

203:

Stanojević, Č. V. On integrability of certain trigonometrical series. Srpska Akad. Nauka. Zb. Rad. **55**, Mat. Inst. **6** (1957), 53–57. (Serbo-Croatian. English summary)

Soit $a_\nu=\alpha_\nu\cdot\beta_\nu=o(1)$, avec $\{\alpha_\nu\}$ à variation bornée, c-à-d. $\sum|\Delta\alpha_\nu|<\infty$, et $\{\beta_\nu\}$ quasi-convexe, c-à-d. $\sum(\nu+1)|\Delta^2\beta_\nu|<\infty$, $|\beta_\nu|\leq M$, alors de $\sum_{\nu=0}^{\infty}|\beta_\nu\Delta\alpha_\nu|\lg(\nu+1)<\infty$ résulte que $a_0/2+\sum_1^\infty a_\nu\cos\nu x$ est la série de Fourier. Pour $\alpha_\nu\equiv1$, respectivement $\beta_\nu\equiv1$, ce résultat se réduit aux théorèmes de Kolmogoroff et Young [A. Zygmund, Trigonometrical series, Warsaw, 1935, p. 109–110]. *M. Tomić* (Belgrade)

204:

González-Fernández, J. M. Integrability of trigonometric series. Proc. Amer. Math. Soc. **9** (1958), 315–319.

The following theorems were established by Heywood [Quart. J. Math., Oxford Ser. (2) **5** (1954), 71–76; MR **16**, 30]. If $\lambda_n\geq0$ ultimately and $g(x)=\sum\lambda_n\sin nx$ then $x^{-\gamma}g(x)\in L$, $1<\gamma<2$, if and only if $\sum n^{\gamma-1}\lambda_n$ converges; the theorem fails for $\gamma\geq2$. If $\lambda_n\geq0$ ultimately, $\frac{1}{2}\lambda_0+\sum_1^\infty\lambda_n=0$, and $f(x)=\frac{1}{2}\lambda_0+\sum_1^\infty\lambda_n\cos nx$, then $x^{-\gamma}f(x)\in L$, $1<\gamma<3$, if and only if $\sum n^{\gamma-1}\lambda_n$ converges; the theorem fails if $\gamma\geq3$. The author shows how, nevertheless, one can extend both theorems to larger values of γ. Let $\lambda_n\geq0$ ultimately; if for some integer $j\geq0$

$$\tfrac{1}{2}\lambda_0+\sum_1^\infty\lambda_n=\sum_1^\infty n^2\lambda_n=\cdots=\sum_1^\infty n^{2j}\lambda_n=0$$

then $x^{-\gamma}f(x)\in L$ if and only if $\sum n^{\gamma-1}\lambda_n$ converges, provided $2j+1<\gamma<2(j+1)+1$; and if and only if $\sum n^{\gamma-1}(\log n)\lambda_n$ converges when $\gamma=2j+1$. If

$$\sum_1^\infty n\lambda_n=\sum_1^\infty n^3\lambda_n=\cdots=\sum_1^\infty n^{2l-1}\lambda_n=0,$$

then $x^{-\gamma}g(x)\in L$ if and only if $\sum n^{\gamma-1}\lambda_n$ converges, provided $2l<\gamma<2(l+1)$; and if and only if $\sum n^{\gamma-1}(\log n)\lambda_n$ converges when $\gamma=2l$. *R. P. Boas, Jr.* (Evanston, Ill.)

205:

Singh, Basudeo. Definite divergence of the conjugate Fourier series. Duke Math. J. **24** (1957), 647–651.

A necessary and sufficient condition is given for the divergence to $\pm\infty$ of the conjugate series of a Fourier series at a given point. The condition is more general than the one given before by Shirley Anderson [Duke Math. J. **14** (1947), 803–805; MR **9**, 140], but the proof runs on familiar lines. There is no reason to believe that technical generalizations like these are in any sense "best-possible". *K. Chandrasekharan* (Bombay)

206:

Izumi, Shin-ichi; and Satô, Masako. Fourier series. VIII. Absolute Riesz summability. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1957), 130–144.

Sufficient conditions are given for the absolute Riesz summability of a Fourier series at a point by the methods $|R,\log n,\delta|$ and $|R,\lambda_n,\delta|$ with $0<\delta<1$. It may be remarked that there is no finality to these results.

K. Chandrasekharan (Bombay)

207:

Ganelius, Tord. Some applications of a lemma on Fourier series. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 9–18.

The lemma in question is as follows: If V is a real integrable function of period 2π and if n is a positive integer, then

$$\sup_t|V(t)|\leq C\Big\{\sum_{m=0}^{n-1}(1-m/n)|V_m|+w(n^{-1};V)\Big\},$$

where C is an absolute constant, V_m are the (complex) Fourier coefficients of V, and $w(\delta;V)$ is the modulus of continuity of V. Among the applications are the following. Th. 1: Let c_m be the Fourier-Stieltjes coefficients of a positive measure μ on the unit circle Γ with $\mu(\Gamma)=1$, let ν be the ordinary arc measure divided by 2π, and let γ be an arc of Γ. Then, for every positive integer n,

$$|\mu(\gamma)-\nu(\gamma)|\leq C\Big\{n^{-1}+\sum_{m=1}^{n-1}(1-m/n)m^{-1}|c_m|\Big\}.$$

Also (Th. 5)

$$|\mu(\gamma)-\nu(\gamma)|\leq C\,|\inf_\Gamma u|^{\frac{1}{2}},$$

where u is the logarithmic potential of μ.

W. Rudin (New Haven, Conn.)

208:

Herz, Carl S. A note on summability methods and spectral analysis. Trans. Amer. Math. Soc. **86** (1957), 506–510.

Sei $Z(f)$ die Menge der Nullstellen der Fouriertransformation $\hat f$ eines $f\in L^1$. Das Spektrum $\Lambda(\phi)$ eines $\phi\in L^\infty$ ist die Menge $\cap Z(f)$, wo der Durchschnitt über alle $f\in L^1$ mit $f\circ\phi\equiv0$ (Faltung) genommen wird. Ist ϕ zudem in L^1, so ist $\Lambda(\phi)$ offenbar der Träger der Fouriertransformation $\hat\phi$. Man kann daher erwarten, daß für die mittels eines Limitierungsverfahrens gewonnene Fouriertransformation $\hat\phi$ eines beliebigen $\phi\in L^\infty$ dasselbe gilt. Beurling [C. R. Acad. Sci. Paris **225** (1947), 326–328; MR **9**, 29] und Pollard [Duke Math. J. **20** (1953), 499–512; MR **15**, 215] benützten dabei das Abel- bzw. das $(R,2)$-Verfahren, während Verfasser nun mit einer allgemeinen Klasse von Verfahren arbeitet und Ergebnisse herleitet, die zum Teil auch in den obengenannten Spezialfällen noch nicht bekannt waren. Seine Verfahren beruhen auf der Bildung von $\Phi_h(t)=\int\exp(-itx)k(hx)\phi(x)dx$ für $h\to0$. Dabei sei k ein Spektralkern (d.h. $k\in L^1$ und $k(0)=1$, ferner gelte eine Darstellung $k(x)=\int_{|x|}^\infty k'(y)dy$ mit $k'\in L^1$). Nach Theorem 1 strebt $\Phi_h(t)\to0$ gleichmäßig in jeder abgeschlossenen Menge mit positivem Abstand von $\Lambda(\phi)$. Die Konvergenzgeschwindigkeit wird dabei noch etwas genauer abgeschätzt. Theorem 2 befaßt sich mit L^2-Konvergenz. In Th. 3 und 4 wird für $\psi=f\circ\phi$ (mit $f\in L^1$) das Verhalten von $\hat f(t)\Phi_h(t)$ und $\Psi_h(t)$ verglichen. Etwas schwieriger sind Umkehrungen der ersten beiden Sätze (aus Eigenschaften von $\Phi_h(t)$ schließe man auf $t\notin\Lambda(\phi)$). Verf. nennt zwei Ergebnisse in dieser Richtung. Beim ersten wird L^1-Konvergenz gegen Null von $\Phi_h(s)$ in einer Umgebung von t vorausgesetzt; beim zweiten zwar nur punktweise Konvergenz, dafür muß dann der Kern die Eigenschaft besitzen, daß bei einem $\phi\in L^\infty$ nur dann $\Phi_h(t)\to0$ für alle t gilt, wenn ϕ fast überall verschwindet.

K. Zeller (Tübingen)

209:

Berg, Lothar. Unharmonische trigonometrische Reihen. Math. Nachr. **16** (1957), 295–308.

The author first considers a sine series of the form

$$(1)\qquad\sum_{\nu=1}^\infty a_\nu\sin k_\nu s,$$

where the a_ν are real, $0<k_1<k_2<\cdots<k_n$, and $k_n\to\infty$ as $n\to\infty$. He obtains various conditions which are necessary and sufficient in order that the series (1) converge to a function $f(s)$ in the following mean square sense:

$$(2)\qquad \lim_{n\to\infty}\int_0^\infty\Big[f(s)-\sum_{\nu=1}^{n}a_\nu\sin k_\nu s\Big]^2\frac{ds}{s^2}=0.$$

By considering sine and cosine transforms, the author is then able to characterize the functions $f(s)$ which can be represented, in the sense of (2), as the sum of the series (1). Analogous problems are treated for the cosine series

$$\sum_{\nu=1}^{\infty}a_\nu\cos k_\nu s,$$

and the paper is concluded with some remarks on approximation and Dirichlet series.

F. W. Gehring (Helsinki)

INTEGRAL TRANSFORMS

See also: 92, 96, 192, 208.

210:

Džrbašyan, M. M. On an integral transform. Izv. Akad. Nauk Armyan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), no. 4, 3–18. (Russian. Armenian summary)

Consider the equation $y''-2ia_0\lambda y'+\lambda^2y=0$, $a_0>0$. A solution $\Phi(x,\lambda)$ with boundary conditions $\Phi(0,\lambda)\cos\alpha+\Phi_x'(0,\lambda)\sin\alpha=0$ is easily obtained, and generates the function $\Psi(x,\lambda)=e^{-2ia_0\lambda x}\Phi(x,\lambda)$, which is used as the kernel of an integral transform, $F(\lambda)=\int_0^\infty f(t)\cdot\Psi(t,\lambda)dt$. The author finds the inverse transform and proves a representation theorem when f is continuous, of bounded variation, and integrable on $(0,\infty)$, and a Plancherel theorem when $f\in L^2$. The inverse is formally

$$f(x)=\pi^{-1}(1+a_0^2)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{F(\lambda)\Phi(x,\lambda)\lambda^2d\lambda}{\lambda^2\sin^2\alpha+\cos^2\alpha+ia_0\lambda\sin 2\alpha}$$

if $\cot\alpha\leqq 0$, while if $\cot\alpha>0$ the term

$$\frac{\cot\alpha}{\sin\alpha(1+a_0^2)^{\frac{1}{2}}}\{\omega_1^2F(-i\omega_1\cot\alpha)+\omega_2^2F(i\omega_2\cot\alpha)\}e^{-x\cot\alpha}$$

has to be added, where $\omega_{1,2}=(1+a_0^2)^{\frac{1}{2}}\pm a_0$.

R. P. Boas, Jr. (Evanston, Ill.)

211:

Kuttner, B. Some theorems on the Cesàro limit of a function. J. London Math. Soc. 33 (1958), 107–118.

Let α be a positive number, not necessarily an integer, and let $\phi(x)$ be Lebesgue integrable over each finite interval $0\leqq x\leqq x_0$. According to familiar definitions, $\phi(x)\to s$ (C,α) as $x\to\infty$ if the first of the relations

$$\lim_{x\to\infty}\frac{\alpha}{x^\alpha}\int_0^x(x-t)^{\alpha-1}\phi(t)dt=s,\quad \lim_{x\to 0+}\frac{\alpha}{x^\alpha}\int_0^x(x-t)^{\alpha-1}\phi(t)dt=s$$

holds, and $\phi(x)\to s$ (C,α) as $x\to 0+$ if the second holds. Another definition says that $\phi(x)\to s$ (N,α) as $x\to\infty$ if $\phi(1/x)\to s$ (C,α) as $x\to 0+$, that is, if

$$(2)\qquad \lim_{x\to\infty}\alpha x\int_x^\infty\frac{(t-x)^{\alpha-1}}{t^{\alpha+1}}\phi(t)dt=s.$$

The integrals are Cauchy extensions of Lebesgue integrals. Two theorems involve the possibility of extending, to Cesàro limits, the simple fact that $\phi(x)\to s$ as $x\to\infty$ if and only if $\phi(1/x)\to s$ as $x\to 0+$. The first theorem says that if $\phi(x)\to s$ (C,α) as $x\to\infty$ and the integral in (2) converges, then $\phi(x)\to s$ (N,α) as $x\to\infty$ provided α is an integer; if α is not an integer, then $\phi(x)\to s$ (N,β) when $\beta>\alpha$ but not necessarily when $\beta=\alpha$. The second theorem says that if $\phi(x)\to s$ (N,α) as $x\to\infty$, then $\phi(x)\to s$ (C,α) as $x\to\infty$ provided α is an integer; if α is not an integer, then $\phi(x)\to s$ (C,β) as $x\to\infty$ when $\beta>\alpha$ but not necessarily when $\beta=\alpha$.

R. P. Agnew (Ithaca, N.Y.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 130, 131, 354, 518, 656, 688.

212:

Pearson, Carl E. On the finite strip problem. Quart. Appl. Math. **15** (1957), 203–208.

The author describes a method of obtaining explicit solutions of integral equations of the form

$$\int_{-1}^{1}[P(x-t)\ln|x-t|+Q(x-t)]f(t)dt=g(x)\quad(a<x<b),$$

when P and Q are polynomials. By use of the Plemelj formulae he obtains an equation of the form

$$\int_x^1P(x-t)f(t)dt=h(x),$$

where $h(x)$ involves $f(x)$ only in terms of the constants

$$c_r=\int_{-1}^{1}t^rf(t)dt\quad(0\leqq r\leqq n+1),$$

n being the greater of the degrees of P and Q. He then obtains a system of linear equations to determine the constants c_r.

Some illustrative examples are given. There is also a set of definite integral formulae that are useful in carrying out transformations of this kind.

F. Smithies (Cambridge, England)

213:

***Pogorzelski, Witold. Równania całkowe i ich zastosowania. Tom 2. Układy równań całkowych; równania całkowe nieliniowe; zastosowania równań całkowych w teorii równań różniczkowych. Z dodatkiem R. Sikorskiego: O twierdzeniu Schaudera. [Integral equations and their applications. Vol. 2. Systems of integral equations; nonlinear integral equations; applications of integral equations to the theory of differential equations. With a supplement by R. Sikorski: On Schauder's theorem.]** Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 192 pp. (1 insert). zł. 24.

[For the review of vol. 1 see MR **16**, 46.] This volume starts with the small Ch. 8 on systems of equations and integro-differential equations. Ch. 9 is concerned with non-linear equations; an existence theorem is given by the method of successive approximations with Lipschitz' condition. In Ch. 10 Banach's contraction theorem is proved and its applicability indicated; then comes the method of Schauder (whose theorem is proved in an appendix). This is used to derive more general existence theorems. An investigation, via Frechet's space, is given of a system with infinitely many unknown functions. The remainder of the book is devoted to connections with ordinary (Ch. 11) and partial differential equations. The topics discussed include: the incorporation of initial

conditions, potentials and quasi-potentials, Green's function and its properties, Dirichlet, Neumann and Cauchy problems, and some special subjects, such as the method of Vekua for elliptic equations with analytic coefficients.

There is an index and a bibliography of 44 titles, most of them referring to the work of Polish and Russian mathematicians. A third volume is promised.

Z. A. Melzak (Montreal, Q.)

214:

Lando, Yu. K. The Cauchy function of a linear integro-differential equation of Volterra type. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 41–47. (Russian)

215:

Lando, Yu. K. On the alternative for certain classes of systems of integro-differential equations. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 49–58. (Russian)

FUNCTIONAL ANALYSIS

See also 82, 109, 159, 338.

216:

Fleischer, Isidore. Maximality and ultracompleteness in normed modules. Proc. Amer. Math. Soc. **9** (1958), 151–157.

(M, R, Λ, V) is a normed module if M is a module with domain of operators R, Λ is a totally ordered set with minimal element θ, and V is a mapping of M onto Λ such that for x, y in M and α in R: (1) $V(x)=\theta$ if and only if $x=0$; (2) $V(x\pm y)\leqq\mathrm{Max}[V(x), V(y)]$; (3) $V(\alpha x)\leqq V(x)$. Gravett [Quart. J. Math., Oxford Ser. (2) **6** (1955), 309–315; MR **19**, 385] shows by means of an embedding theorem that each normed module M has a unique maximal immediate extension provided that R is a field. The author gives an example of a normed module with at least two non-isomorphic maximal immediate extensions. Let $M_\lambda=\{x\in M: V(x)\leqq\lambda\}$. M is ultra-complete if it is complete in every topology obtained by taking as a basis at 0 any subfamily of the M_λ. M is ultra-complete if and only if M is pseudo-complete (every pseudosequence has a pseudo-limit). It is known [loc.cit.] that if R is a field, then M is pseudo-complete if and only if M admits no (proper) immediate extensions. The author shows that an ultra-complete module has no immediate extensions, and conversely if M has no immediate extensions and R is a principal ideal ring, then M is ultra-complete.

$(M, K, \Lambda, V, \Gamma, v)$ is a normed space if K is a field, (M, K, Λ, V) is a normed module, v is a valuation of K with value group Γ, Γ acts on Λ, and for $\lambda, \mu\in\Lambda$, $\alpha\in K$, $\gamma, \delta\in\Gamma$ and $x\in M$: (1); (2); (3′) $V(\alpha x)=v(\alpha)V(x)$; (4) $(\gamma\delta)\lambda=\gamma(\delta\lambda)$; (5) $\gamma\leqq\delta$ implies $\gamma\lambda\leqq\delta\lambda$; (6) $\lambda\leqq\mu$ implies $\gamma\lambda\leqq\gamma\mu$. Each normed space admits a unique maximal immediate extension. An embedding theorem is obtained and this leads to a structure theorem for normed spaces. These results can be applied to ordered vector spaces.

P. F. Conrad (New Orleans, La.)

217:

Yamamuro, Sadayuki. Monotone completeness of normed semi-ordered linear spaces. Pacific J. Math. **7** (1957), 1715–1725.

A norm on a continuous (σ-complete) normed semi-ordered linear space R is said to be monotone complete if every norm-bounded monotone sequence of elements is order-bounded. The author shows that the monotone completeness is equivalent to the completeness of the norm together with the K-boundness (first considered by L. V. Kantorovitch), i.e., a lattice theoretical property of the space R as follows: every sequence of elements x_ν ($\nu=1, 2, \cdots$) such that $\xi_\nu x_\nu$ are order-bounded for every sequence of real numbers ξ_ν converging to 0, is itself order-bounded.

He also gives several results concerning the relations of the monotone completeness to the continuity of the norm and, when the norm of R is defined by a modular, to the properties of this modular.

I. G. Amemiya (Kingston, Ont.)

218:

Tatarkiewicz, Krzysztof. Une démonstration du théorème de Frink généralisé. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 33–36 (1958). (Polish and Russian summaries)

A simpler proof of a theorem in the author's paper in Ann. Univ. Mariae Curie-Skłodowska. Sect. A **6** (1952), 47–54 (1954); MR **16**, 832.

219:

Foguel, Shaul R. On a theorem by A. E. Taylor. Proc. Amer. Math. Soc. **9** (1958), 325.

The reviewer proved for a normed linear space B that if the unit sphere in the conjugate space is strictly convex, then every continuous linear functional on an arbitrary linear subspace of B has a unique extension without increase of norm to all of B. He also shows that the converse holds when B is reflexive. [A. E. Taylor, Duke Math. J. **5** (1939), 538–547; MR **1**, 58]. In this paper the converse theorem is proved without the reflexivity hypothesis. In fact, if f_1 and f_2 are unequal elements of unit norm in the conjugate space and if $\|f_1+f_2\|=2$, then f_1 and f_2 are both norm-preserving extensions of their common restriction to the subspace on which f_1 and f_2 coincide.

A. E. Taylor (Los Angeles, Calif.)

220:

Kurzweil, J. On approximation in real Banach spaces by analytic operations. Studia Math. **16** (1957), 124–129.

Let B and B_1 be Banach spaces of which the first is real. The author has shown earlier [Studia Math. **14** (1954), 214–231; MR **16**, 932] that if B is separable and satisfies (A) there exists a real polynomial $q^*: B\to B_1$ such that $q^*(\theta)=0$ and

$$\inf\{q^*(x): x\in B, \|x\|=1\}>0,$$

then every continuous operation $F: B\to B_1$ is the uniform limit (on balls) of analytic operations. The present paper shows that if B is uniformly convex, then (A) is necessary in order that such an approximation be possible. This results from the following main Th. 1: Suppose B is uniformly convex and that

$$\inf\{|q(x)-q(\theta)|: x\in B, \|x\|=1\}=0$$

for every real polynomial q. If f is real and analytic for $\|x\|<R$, and if $0<\varepsilon<r<r+\varepsilon<R$, there exists an $x\in B$ satisfying $r\leqq\|x\|<r+\varepsilon$ and $|f(x)-f(\theta)|<\varepsilon$.

R. E. Edwards (Woking)

221:

Pełczyński, A. A property of multilinear operations. Studia Math. **16** (1957), 173–182.

Some negative results of Kurzweil [Studia Math. **14** (1954), 214–231; MR **16**, 932], concerning the impossibility of polynomial approximation to the norm on certain

normed vector spaces, are here obtained by an independent approach. E and E_1 being normed vector spaces, a concept of sequential convergence in E is introduced: a sequence $(x_n) \subset E$ is said to be τ_s-convergent to Θ (zero element of E) if there exists a constant C such that

$$\|\varepsilon_1 x_{n_1} + \cdots + \varepsilon_k x_{n_k}\| \leq C k^s$$

for arbitrary distinct indices $n_1, \cdots, n_k$ and arbitrary ± 1 sequences (ε_i); here $0 \leq s < 1$; (x_n) is τ_s-convergent to x if $(x_n - x)$ is τ_s-convergent to Θ. One then considers the τ_s-continuity of operations $A: E^r \to E_1$ (using the normed topology on E_1). E_1 is said to have rank (resp. loose rank) s if its norm is τ_α-continuous for $0 \leq \alpha \leq s$ (resp. $0 \leq \alpha < s$). The main Theorem 1 asserts that if $A: E^r \to E_1$ is r-linear, and if E_1 has rank (resp. loose rank) s_1, then A is τ_s-continuous if $rs \leq s_1$ (resp. $rs < s_1$). Applications are given for the cases in which $E = l^p$, $E_1 = l^q$, $E_1 = K^m$ (K = scalar field). Moreover, each weakly complete space is shown to have rank 0, so that any r-linear operation: $E^r \to$ (a weakly complete space) is τ_0-continuous.

If it be supposed that in E there exist essentially τ_0-convergent sequences (i.e. ones which are τ_0-convergent but not convergent in norm) then (Theorem 2) the norm on E cannot be approximated by polynomials on any ball in E. The space $E = C(Q)$ (Q an infinite compact metric space) contains always essentially τ_0-convergent sequences. The same is true of the space $c\langle 0, 1\rangle$ of real functions x on $\langle 0, 1\rangle$ such that for every $\delta > 0$ the set $\{t \in \langle 0, 1\rangle : |x(t)| \geq \delta\}$ is finite.

{The paper contains a few minor misprints: in Th. 1, "r_1" should read "s_1"; in line 13, p. 178, "l^p" should read "l^q"; and in Th. 3, "functionals" should read "polynomials".} *R. E. Edwards* (Woking)

222:

Orlicz, W.; and Ciesielski, Z. Some remarks on the convergence of functionals on bases. Studia Math. **16** (1958), 335–352.

The authors consider for the most part Banach spaces, X, but (F) spaces are also treated. A subset B of the set $A \subset X$ is called a linear rational basis (LRB) of A if $x \in A$ implies $x = \sum_{i=1}^{n} \alpha_i x_i$ with $x_i \in B$ and α_i rational. A subset B^* of $A \subset X$ is a convex rational basis (CRB) of A if there exists $a \in B^*$ and an $M > 0$ such that $x \in A$ implies $x = \sum_{i=1}^{m} \beta_i (x_i - a) + a$ where $x_i \in B^*$, $\beta_i \geq 0$, β_i is rational, $\beta_1 + \cdots + \beta_m \leq M$, m depending on x. A functional ξ defined in a convex set D is called convex if $\xi(\lambda x + \mu y) \leq \lambda \xi(x) + \mu \xi(y)$ for λ, μ rational satisfying $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu = 1$. The authors prove a substantial number of results on sequences of linear functionals to the effect that behavior of a proper kind of the sequence on an LRB or CRB of A ensures the corresponding behavior over A. The examples given below will serve to give meaning to this statement.

Th.: Let ξ_n be a functional continuous and convex in D for $n = 1, 2, \cdots$ and let $\{\xi_n\}$ be convergent (bounded) in a set dense in D. Then the sequence $\{\xi_n\}$ is either convergent (locally uniformly bounded) in D or convergent on (bounded above in) a set which is not a CRB of D. The Banach-Steinhaus theorem receives this generalization: A sequence of linear operations $\{U_n\}$ is either uniformly bounded in the unit sphere or bounded in a set which is not an LRB of X.

The class S_k^* is defined to consist of those functionals ξ such that for arbitrary $x_1, \cdots, x_{k+1}$, $\xi(x_1 + \cdots + x_{k+1}) \leq \xi(x_1) + \cdots + \xi(x_{k+1})$. Th.: Suppose the functional ξ_n to be continuous on X, $\xi_n \in S_k^*$ and $\xi_n \leq \xi_{n+1}$, $n = 1, 2, \cdots$. If the sequence ξ_n is convergent in a set dense in X and in a CRB of a sphere in X, then it is convergent in the whole space X. Various applications of these results are given. Th.: If $E \subset A \subset X$, A being a bounded set, and if E is of the second category and satisfies the condition of Baire, then E is a CRB of A. Th.: An arbitrary measurable set $E \subset E_n$ of positive measure contained in a sphere $K(x_0, r)$ is a CRB of this sphere. Th.: If the series

$$\sum_{n=1}^{\infty} (|a_n \cos \lambda_n x| + |b_n \sin \lambda_n x|), \quad |\lambda_n| \to \infty,$$

is convergent in an LRB of an interval (α, β), then $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$. *E. R. Lorch* (New York, N.Y.)

223:

Sz.-Nagy, Béla. Suites faiblement convergentes de transformations normales de l'espace hilbertien. Acta Math. Acad. Sci. Hungar. **8** (1957), 295–302.

Let H be a complete inner-product space of any dimension and let B be the algebra of bounded linear operators on H. If $E \subset B$, let $\bar{E}$ be the weak closure of E and $\hat{E}$ be the set of limits of weakly convergent sequences from E. Von Neumann [Math. Ann. **102** (1929), 370–427] showed that generally $\hat{E}$ is smaller than $\bar{E}$. Let E_T be the set of all operators of norm ≤ 1, let E_U be the set of unitary operators, let E_A be the set of operators between 0 and 1, and let E_P be the set of orthogonal projections in B. Halmos [Summa Brasil. Math. **2** (1950), 125–134; MR **13**, 359] proved that $E_T = \bar{E}_U$ and $E_A = \bar{E}_P$. The author proves in this note that $E_T = \hat{E}_U$ and $E_A = \hat{E}_P$; he gives some extensions of this and applications to subnormal operators and *-semigroups of operators.

M. M. Day (Urbana, Ill.)

224:

Koosis, Paul. Interior compact spaces of functions on a half-line. Comm. Pure Appl. Math. **10** (1957), 583–615.

Dans l'espace $L^2(0, +\infty)$, considérons les opérateurs $f \to f_h$, h réel ≥ 0, avec $f_h(x) = f(x+h)$, $x \geq 0$. Un sous-espace vectoriel fermé E de $L^2(0, +\infty)$ est dit "intérieur compact", si tous les opérateurs précédents laissent E stable et sont complètement continus sur E. L'auteur montre d'abord que tout espace E intérieur compact est engendré par les exponentielles-monômes qu'il contient (c.à.d. les fonctions $x^k \exp(i\lambda x)$, k entier ≥ 0, $\operatorname{Im} \lambda > 0$). Le spectre Λ de E est l'ensemble des λ correspondants, λ étant multiple d'ordre $p+1$ dans Λ si $x^k \exp(i\lambda x) \in E$ exactement pour $k \leq p$. Réciproquement soit Λ un ensemble de nombres complexes λ_n, $\operatorname{Im} \lambda_n > 0$, λ_n multiple d'ordre $p_n + 1$; soit E le sous-espace fermé engendré par les $x^k \exp(i\lambda_n x)$, $\lambda_n \in \Lambda$, $k \leq p_n$. Quelle est la condition (Γ) nécessaire et suffisante, relative à Λ, pour que E soit intérieur compact? C'est la suivante:

$$\text{(a)} \qquad \lim_{n \to \infty} \operatorname{Im} \lambda_n = +\infty,$$

$$\text{(b)} \qquad \lim_{\sigma \to \pm\infty} \sum_{n} \frac{(p_n + 1) \operatorname{Im} \lambda_n}{|\lambda_n - \sigma|^2} = 0.$$

Dans une première partie, l'auteur donne une condition nécessaire plus faible que (Γ); dans la deuxième partie il démontre que (Γ) est suffisante. Ce n'est que plus tard qu'il a pu montrer que la condition (Γ) était aussi nécessaire; c'est pourquoi la nécessité est seulement démontrée dans un addendum à la fin de l'article. Entre temps Lax avait montré que la condition était nécessaire et suffisante, par une méthode assez différente, donnée dans un article qui suit celui-ci dans la même revue [≠ 225 ci-dessous].

Les démonstrations utilisent, comme toujours dans ce

genre de problèmes, les produits de Blaschke, et des intégrales de contour dans le plan complexe pour exprimer et majorer des fonctions analytiques liées à ces produits. Comme conséquence de la condition nécessaire et suffisante, on voit que le sous-espace fermé engendré par la somme de deux sous-espaces intérieurs compacts est aussi intérieur compact. *L. Schwartz* (Paris)

225:

Lax, P. D. Remarks on the preceding paper. Comm. Pure Appl. Math. **10** (1957), 617–622.

Voir l'article de Koosis [# 224 ci-dessus]. *L. Schwartz* (Paris)

226:

de Barros Neto, J. Sur les distributions et les intégrales de Cauchy. Bol. Soc. Mat. São Paulo **9** (1954), 72–80 (1957).

Dans le plan complexe $z=x+iy$, on a [cf. L. Schwartz, Théorie des distributions, Hermann, Paris, 1950–51, MR 12, 31, 833]:

$$\frac{\partial}{\partial\bar{z}}\left(\text{v.p.}\,\frac{1}{z^m}\right)=(-1)^{m-1}\,\frac{\pi}{(m-1)!}\,\frac{\partial^{m-1}}{\partial z^{m-1}}\,\delta.$$

L'A. donne quelques applications de cette formule. Généralisation au cas des dimensions quelconques. *J. L. Lions* (College Park, Md.)

227:

Fantappiè, Luigi. Sull'integrale affine di una funzione analitica di due *n*uple di variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 539–547.

È questo l'ultimo lavoro di carattere matematico di Luigi Fantappiè, ordinario dell'Istituto di Alta Matematica dell'Università di Roma, deceduto il 28 luglio 1956. Notizie sull'opera complessiva del Fantappiè possono trovarsi sia nel mio articolo, Rend. Mat. e Appl. (5) **15** (1956), 505–519 [MR **19**, 108], sia nella magistrale commemorazione tenuta da G. Fichera all'Istituto di Alta Matematica [ibid. **16** (1957), 143–160; MR **19**, 1248].

In quest'ultimo lavoro il Fantappiè si occupa ancora della sua teoria dei funzionali analitici [esposta nelle sue grandi linee nella IV parte, da me redatta, del volume di P. Lévy, Problèmes concrets d'analyse fonctionnelle, 2ième éd., Gauthier-Villars, Paris, 1951; MR **12**, 834] e piú precisamente dei funzionali analitici delle funzioni di più variabili. Strumenti notevoli nello studio di tali funzionali — aventi del resto interesse di carattere generale — sono la "traccia proiettiva" e la formazione puramente integrale, ora introdotta, detta "integrale affine".

Della traccia proiettiva $\overset{\Delta}{T}q$ di una funzione q di due n-uple $\alpha_1, \cdots, \alpha_n$ e $t_1, \cdots, t_n$, regolare in due n-cilindri Q e Q' con raggi reciproci si ha, sia l'espressione

$$\overset{\Delta}{T}q=\frac{1}{(2\pi i)^n}\int_{C_1}d\alpha_1\cdots\int_{C_n}d\alpha_n\int_\Omega d\omega\,\frac{\partial^{n-1}}{\partial t_{1}^{n-1}}$$

$$\times\left\{\frac{t^{n-1}}{\alpha_1\cdots\alpha_n}\,q\left(t\alpha_1,\cdots,t\alpha_n,\frac{-\tau_1}{\alpha_1},\frac{\tau_1-\tau_2}{\alpha_2},\cdots,\frac{\tau_{n-1}-1}{\alpha_n}\right)\right\}$$

(ove Ω è la piramide $0\leqq\tau_1\leqq\cdots\leqq\tau_{n-1}\leqq 1$) come pure le espressioni che si ottengono permutando a nostro arbitrio gli n numeratori, dipendenti dalle variabili τ_s.

Se la funzione q si spezza nel prodotto di due funzioni $p(\alpha_1,\cdots,\alpha_n)$ e $y(t_1,\cdots,t_n)$ delle due $n$uple separate, la traccia proiettiva di questo prodotto si riduce al prodotto funzionale proiettivo $p\nabla y$ delle due funzioni

$$\overset{\Delta}{T}\{p(\alpha_1,\cdots,\alpha_n)y(t_1,\cdots,t_n)\}=p(\alpha_1,\cdots,\alpha_n)\nabla y(t_1,\cdots,t_n)$$

che è una formazione molto importante, perché ogni funzionale analitico lineare $F[y(t_1,\cdots,t_n)]$ definito per i polinomi può rappresentarsi in questo modo, se y è regolare in Q' e se la sua indicatrice proiettiva p, definita dalla formula

$$p(\alpha_1,\alpha_2,\cdots,\alpha_n)=F\left[\frac{1}{1+\alpha_1t_1+\cdots+\alpha_nt_n}\right]$$

è regolare in Q.

Introdotta "l'indicatrice affine" del funzionale lineare F

$$p_1(\alpha_1,\alpha_2,\cdots,\alpha_n)=\frac{\partial^{n-1}}{\partial t_{1}^{n-1}}\{t^{n-1}p(t\alpha_1,t\alpha_2,\cdots,t\alpha_n)\}$$

$$=\frac{\partial^{n-1}}{\partial t_{1}^{n-1}}\,F\left[\frac{t^{n-1}}{1+t(\alpha_1t_1+\alpha_2t_2+\cdots+\alpha_nt_n)}\right]$$

$$=(n-1)!F\left[\frac{1}{(1+\alpha_1t_1+\cdots+\alpha_nt_n)^n}\right],$$

si ha

$$F[y]=p\nabla y=\overset{\Delta}{T}\{py\}=$$

$$\frac{1}{(2\pi i)^n}\int_{C_1}d\alpha_1\cdots\int_{C_n}d\alpha_n\int_\Omega d\omega p_1(\alpha_1,\cdots,\alpha_n)$$

$$\times\,\frac{1}{\alpha_1\cdots\alpha_n}\,y\left(\frac{-\tau_1}{\alpha_1},\frac{\tau_1-\tau_2}{\alpha_2},\cdots,\frac{\tau_{n-1}-1}{\alpha_n}\right).$$

L'integrale affine di q può essere definito a mezzo dell'espressione

$$\int^\Delta qd\alpha=\frac{1}{(2\pi i)^n}\int_{C_1}d\alpha_1\cdots\int_{C_n}d\alpha_n\int_\Omega d\omega$$

$$\times\,\frac{1}{\alpha_1\cdots\alpha_n}\,q\left(\alpha_1,\cdots,\alpha_n,\frac{-\tau_1}{\alpha_1},\frac{\tau_1-\tau_2}{\alpha_2},\cdots,\frac{\tau_{n-1}-1}{\alpha_n}\right).$$

Esso dipende simmetricamente dalle due $n$uple di variabili, il suo valore non cambia per sostituzioni lineari omogenee "duali" sulle due $n$uple di variabili (contragredienti), usando dell'integrale affine il Fantappiè obtiene

$$F[y]=p\nabla y=\overset{\Delta}{T}\{p(\alpha_k)y(\alpha^k)\}=$$

$$\int^\Delta p_1(\alpha_1,\cdots,\alpha_n)y(\alpha^1,\cdots,\alpha^n)d\alpha$$

e da questa la notevole formula integrale per il valore di una funzione in un punto

$$(*)\quad y(z_1,\cdots,z_n)=$$

$$\frac{(n-1)!}{(2\pi i)^n}\int_{C_1}d\alpha_1\cdots\int_{C_n}d\alpha_n\int_\Omega d\omega\,\frac{1}{\alpha_1\cdots\alpha_n}$$

$$\times\frac{1}{(1+\alpha_1z_1+\cdots+\alpha_nz_n)^n}\,y\left(\frac{-\tau_1}{\alpha_1},\frac{\tau_1-\tau_2}{\alpha_2},\cdots,\frac{\tau_{n-1}-1}{\alpha_n}\right),$$

data anche, sotto forma diversa, dal Leray. La possibilità di ottenere una tale formula dai risultati precedenti del F era prevedibile. Io stesso alcuni anni fa l'avevo assegnata come tema di ricerca ad un discepolo ricercatore dell'Istituto di Alta Matematica.

Se $n=2$ e $y(t_1,t_2)=(Q(t_1,t_2))^{-1}$, con Q polinomio, il Fantappiè trova

$$\int^\Delta\frac{1}{Q(\alpha^1,\alpha^2)}p_1(\alpha_1,\alpha_2)d\alpha=\frac{1}{2\pi i}\iint_V\frac{1}{J(x_1,x_2)}p_1(x_1,x_2)dx_1dx_2$$

formula che nel caso della cardioide

$$Q(\alpha_1,\alpha_2)=4+12\alpha_1-15\alpha_1^2-27\alpha_2^2-4\alpha_1^3=0,$$

ove si specializzi ϕ_1 nell'indicatrice affine del funzionale analitico lineare

$$F[y(t_1, t_2)]=y(z_1, z_2),$$

permette di dare il caso particolare della (*),

$$(**)\quad \frac{1}{Q(z_1, z_2)}=\frac{1}{2\pi i}\iint_V \frac{1}{(1+x_1z_1+x_2z_2)^2}\frac{1}{J(x_1, x_2)}dx_1dx_2 + \frac{1}{36\sqrt{3}}\int_{-\sqrt{(3/3)}}^{+\sqrt{(3/3)}} \frac{1}{(1-\frac{1}{2}z_1+x_2z_2)^2}dx_2,$$

con cui questa funzione razionale viene decomposta in due parti (razionali), una data dall'integrale doppio, delimitato dalla cardioide, l'altra data dall'integrale semplice, lungo la retta doppia dell'inviluppo, tra i punti di contatto.

Queste parti sono dunque relative alle due parti, in cui si spezza la curva singolare della funzione, pensata però non nel piano α_1, α_2 iniziale (rigato), ma nel piano duale x_1, x_2, e cioè come luogo dei punti di contatto, curva che si spezza appunto nella cardioide e nella retta doppia, contata due volte.

Val la pena di riportare per esteso le frasi con cui il F. termina: "Volendo quindi ricercare una possibile decomposizione delle funzioni analitiche di due variabili in parti "principali" relative alle singole curve (analitiche), singolari per una tale funzione, mediante la formula (*), o quella di Leray, vediamo che il caso particolare (**) ci indica un fatto importantissimo, e cioè che le singole curve singolari della funzione analitica $y(z_1, z_2)$ vanno cercate non nel piano iniziale (z_1, z_2), ma nel piano duale (x_1, x_2), e cioè nella decomposizione della curva luogo dei punti di contatto dell'inviluppo delle rette, che hanno coordinate (di "retta") z_1, z_2, singolari per la y. Ciò può forse contribuire a spiegare la radicale diversità fra le proprietà delle funzioni analitiche di una variabile, che sono definite su una retta, il cui spazio duale coincide con la retta stessa, e quelle delle funzioni analitiche di due o più variabili, che sono invece definite su uno spazio punteggiato, ben diverso dal corrispondente spazio duale (rigato o planato)".

F. Pellegrino (Roma)

228:

Fišman, K. M. On the representation of certain classes of analytical functions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 466–469. (Russian)

The author considers two separable proper functional Hilbert spaces, H and $\mathscr{H}$ with functions $f(z)$ $(z \in G)$ and $\varphi(\zeta)$ $(\zeta \in \mathscr{G})$ respectively. Theorem 1 concerns a representation of a linear isometry V of H onto $\mathscr{H}$ by means of a kernel $k(z, \zeta)=\sum p_n(z)\overline{q_n(\zeta)}$ where $\{p_n(\zeta)\}$ is an orthonormal basis of H, $q_n(\zeta)=Vp_n$ is the corresponding basis of $\mathscr{H}$. If $\varphi=Vf$, then $\varphi(\zeta)=(f(z), k(z, \zeta))_H$, $f(z)=(\varphi(\zeta), \overline{k(z, \zeta)})_{\mathscr{H}}$. Theorem 2 concerns a proper functional Hilbert space H imbedded in a larger (not necessarily proper functional) Hilbert space $\tilde{H}$ and the statement is made that the reproducing kernel of H represents the projection on H in $\tilde{H}$. {The author mistakenly affirms that this statement is more general than the one of the reviewer on p. 345 of Trans. Am. Math. Soc. **68** (1950), 337–404 [MR **14**, 479].} These theorems allow one to construct explicit representations of functions in one proper functional Hilbert space by functions in another Hilbert space. In the remainder of the paper the author gives several examples of such representations, especially for spaces of analytic functions in the plane. He does not indicate any application of such representations to the study of functions in the spaces concerned.

N. Aronszajn (Lawrence, Kans.)

229:

Lax, P. D. A Phragmén-Lindelöf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations. Comm. Pure Appl. Math. **10** (1957), 361–389.

Let $\overline{R^+}$ be the semi-axis $x \geq 0$ and B a Banach space. Denote by $\mathscr{L}^p(B)$ $(1 \leq p \leq \infty)$ all locally integrable functions on $\overline{R^+}$ with values in B. We write $L^p(B)$ for the corresponding global spaces. Let $\mathscr{A} \subset \mathscr{L}^p(B)$ be (right) translation invariant, and closed in the topology defined by L^p convergence on compact subsets of $\overline{R^+}$. We say that $\mathscr{A}$ satisfies the Phragmen-Lindelöf principle (PL) if, for any $f \in \mathscr{A}$, either $\|f\|=\|f\|_\infty=(\int_0^\infty \|f(x)\|^p dx)^{1/p}=\infty$, or $(\int_0^\infty \|f(x)e^{\alpha x}\|^p dx)^{1/p}<\infty$, where $\alpha>0$ only depends on $\mathscr{A}$. Introduce $A=\mathscr{A} \cap L^p(B)$. It is complete in the $\|\cdot\|$ topology and thus a Banach space, where the translation operators $T(\eta)$ are continuous. Then (PL) is equivalent to (PL') $\|T(\eta)\| \leq Ce^{-\alpha\eta}$. Call $\mathscr{A}$ (A) interior compact (IC) if the set of all f with $\|f\|_{a^b} \leq 1$ is precompact with respect to the seminorm $\|f\|_{a'^{b'}}$ for $a<a'<b'<b$.

Th. 1: If $\mathscr{A}$ is IC, then (PL) holds true. This is a generalization of a well-known theorem on harmonic functions, and it gives a concrete theorem on the growth of certain solutions of elliptic equations in half-cylinders, when B is taken to be the Hilbert space defined by the Dirichlet integral over the base of the cylinder. It also contains certain results on the spaces considered in the thesis of Schwartz [Etude des sommes d'exponentielles réelles, Hermann, Paris, 1943; MR **7**, 294]. The proof is based on Th. 2: $A=\mathscr{A} \cap L^p(B)$ is IC if and only if $T(\eta)$ is compact for $\eta>0$.

The principal difficulty is to show that the condition is necessary. Let A be IC. Then Th. 3: The eigenfunctions of $T(\eta)$ are complete if B is finite dimensional.

This is proved by means of a theorem of Beurling [Math. Scand. **1** (1953), 127–130; MR **15**, 234]. Still more questions are treated in this interesting paper, but it is not possible to give a full account of all problems treated. {Reviewer's remark. Let $\mathscr{C}$ be the topology defined by L^p-convergence on compact subsets of R^+. Then $\mathscr{A}_\mathscr{C}$ is a locally convex space and it is easy to see that $\mathscr{A}$ is IC if and only if $\mathscr{A}_\mathscr{C}$ is a Schwartz space [Grothendieck, Mem. Amer. Math. Soc. no. **16** (1955); Segundo symposium sobre algunos problemas matemàticos que se están estudiando en Latino América, 1954, Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954, pp. 173–177; MR **17**, 763, 765]. If we require $\mathscr{A}_\mathscr{C}$ to be nuclear ($\mathscr{A}$ is then said to be interior nuclear (IN)), then the results above may be sharpened. For instance, we may show that $\|T(\eta)\|_{\mathrm{Tr}} \leq C_{\eta_0}e^{-\alpha\eta}$, $\alpha>0$, $\eta \geq \eta_0>0$, where $\|\cdot\|_{\mathrm{Tr}}$ means the trace norm. Further, the values of $f \in \mathscr{A}$ at $x \in R^+$ are given by a reproducing kernel. It is also possible to prove Theorem 2 with "compact" replaced by "nuclear". It is interesting to note that the spaces considered by Schwartz [loc.cit.] are IN.}

J. E. Roos (Paris)

230:

Fišman, K. M.; and Valickiĭ, Yu. N. The applicability of Fredholm's theory to certain linear topological spaces, Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 943–946. (Russian)

For each r with $\alpha<r \leq \beta$ let $\mathscr{B}_r$ be a Banach space, $\mathscr{B}_+^*$ its dual. Suppose that for $r'<r$ the space $\mathscr{B}_r$ is a dense subspace of $\mathscr{B}_{r'}$ and $\|f\|_{r'} \leq \|f\|_r$. Let $\mathscr{A}_r=\bigcap_{r'<r}\mathscr{B}_{r'}$ and $\mathscr{A}_r^*=\bigcup_{r'<r}\mathscr{B}_{r'}^*$. $\mathscr{A}_r$ and $\mathscr{A}_r^*$ can be topologized so that $\mathscr{A}_r^*$ becomes the

dual of $\mathscr{A}_r$. Let A be a linear operator mapping each $\mathscr{B}_r$ into itself. Then A defines a linear operator in each $\mathscr{A}_r$ and a dual A^* mapping each $\mathscr{B}_r^*$ into itself and each $\mathscr{A}_r^*$ into itself. Assuming that the usual Fredholm relations are satisfied in $\mathscr{B}_r$ and $\mathscr{B}_r^*$ for all r the author shows that they are also satisfied in $\mathscr{A}_r$ and $\mathscr{A}_r^*$. As an application he discusses the special case in which $\mathscr{B}_r$ is the set of all functions of the form $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty}|a_n|^p\alpha_n(r)<\infty$. Here each $\alpha_n(r)$ is a fixed monotone function of r, and $p\geqq 1$. Some results of Evgrafov [Trudy Moskov. Mat. Obšč. **5** (1956), 89–201; MR **18**, 472] are obtained by specializing further.

G. W. Mackey (Cambridge, Mass.)

231:

Leĭbenzon, Z. L. On estimation of the eigenvalues of self-adjoint operators. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 371–373. (Russian)

The author considers a self adjoint operator A in a Hilbert space H with discrete spectrum $\{\lambda_i\}$ and corresponding eigenvectors e_i. For a nonzero element $x=\sum x_i e_i$ he considers the class M of all functions $F(\lambda)$ for which $\|F\|^2=\sum|f(\lambda_i)|^2|x_i|^2<\infty$. With this norm M becomes a Hilbert space and the author's fundamental lemma is: For F and Φ in M and $\|\Phi\|>0$, there exists always an eigenvalue λ_i such that $\Phi(\lambda_i)\neq 0$ and $|F(\lambda_i)|/|\Phi(\lambda_i)|\leqq\|F\|/\|\Phi\|$. The author applies the lemma to the case when $A^n x$ is defined; hence when all polynomials of order $\leqq n$ belong to M and taking $f(\lambda)=(\lambda-\alpha)(\Phi(\lambda))$ and Φ a polynomial of order $\leqq n-1$, he obtains the theorem: There exists an eigenvalue in the interval $\alpha-(\|F\|/\|\Phi\|)\leqq\lambda\leqq\alpha+(\|F\|/\|\Phi\|)$. For $n=1$, this inequality was proved by T. Kato [J. Phys. Soc. Japan **4** (1949), 334–339; MR **12**, 447]. To obtain better evaluations he considers the orthonormal polynomials $D_m(\lambda)$ corresponding to the moment problem with masses $|x_i|^2$ at λ_i. He then chooses $F(\lambda)=D_m(\lambda)$, α being a root of $D_n(\lambda)$. Replacing x by $y^t=A^t x$ he obtains the corresponding polynomials $D_{m,t}(\lambda)$ and states the theorem: If A is bounded then the largest m roots, $r_{1,t}\cdots r_{m,t}$ of $D_{m,t}(\lambda)$ converge to the m largest eigenvalues λ_i for $t\to\infty$. Moreover, $|r_{i,t}-\lambda_i|\leqq C(\gamma/\lambda_i)^{2t}$, $i\leqq m$, and $\gamma=\sup_{j>m}|\lambda_j|$. Here C is independent of t {but C depends on the masses $|x_i|^2$ and the theorem is obviously not valid if any one of the x_i's, $i\leqq m$, is zero (Reviewer's remark)}.

N. Aronszajn (Lawrence, Kans.)

232:

Markova, V. A. On linear operator equations with an operator depending on a parameter. Penzen. Gos. Ped. Inst. V. G. Belin. Uč. Zap. **4** (1956), 111–123. (Russian)

233:

Putnam, C. R. On semi-normal operators. Pacific J. Math. **7** (1957), 1649–1652.

Let A be a bounded operator in a Hilbert space, $J_\varphi=\frac{1}{2}(Ae^{-i\varphi}+A^*e^{i\varphi})$ for real φ, and $H=AA^*-A^*A$. The author calls A semi-normal if $H\geqq 0$. He shows that if A is semi-normal and $\lambda=re^{i\varphi}$ is a boundary point of the spectrum of A, then $R_\varphi\equiv(\max J_\varphi)^2-\min AA^*\geqq 0$ and $|r-\max J_\varphi|\leqq R_\varphi^{\frac{1}{2}}$. This implies that if $\lambda=0$ is in the spectrum of A and $\min AA^*>0$, then the spectrum of A contains, for every φ, the disk $|\lambda|\leqq(\max J_\varphi)-R_\varphi^{\frac{1}{2}}$. It is also shown that if $J_\varphi=\int\lambda dE$ and S is a measurable set such that $\int_S dE=I$, then $\|H\|\leqq 4\|A\|\text{meas } S$.

P. Hartman (Baltimore, Md.)

234:

Kaazik, Yu. J. On a class of iteration processes for the solution of the operation equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 579–582. (Russian)

The author describes two fixed point theorems in normal linear spaces for iterative solutions of the equation

$$P(x)=0,$$

where P is a twice-differentiable operator from a Banach space X into a normal space Y, by means of the iteration

$$x_{n+1}=x_{n+1}-x_n=-(E+\alpha R_n)^{-1}[E+(\alpha+1)R_n]\,\Gamma_n P(x_n),$$

where $\Gamma_n=[P'(x_n)]^{-1}$, $R_n=\frac{1}{2}\Gamma_n P''(x_n)\Gamma_n P(x_n)$, E is the identity, and α a real number. For $\alpha=0$, the "Chebychev process" is obtained, the case $\alpha=1$ is a process of "tangent hyperbolas" discussed by Mertvetcova [same Dokl. **88** (1953), 611–614; MR **15**, 39], and the case $\alpha=-2$ is given in a dissertation of Vykhandy at the Tartar University. The first theorem assumes analyticity of P within a sphere around the initial guess and produces bounds for the rate of convergence to a solution; the second theorem assumes differentiability of P through the third order in a sphere about the initial guess and produces bounds for the rate of convergence to a solution. The proof of the first theorem is given; the proof of the second theorem is stated to be similar to that of Kantorovich for Newton's method.

J. W. Carr, III (Ann Arbor, Mich.)

235:

Povolockiĭ, A. I. Application of a variational method to investigation of the spectra of non-linear operators. Mat. Sb. N.S. **42(84)** (1957), 287–300. (Russian)

This paper proves the existence of characteristic vectors on spheres for certain non-linear operators Γ in Hilbert space H. It is assumed first that Γ is the gradient of some weakly differentiable function F and second that there is a self-conjugate, completely continuous linear operator B such that on some sphere $S=\{x:\|x\|=\rho\}$ it is known that $\|\Gamma x-Bx\|\leqq K\|x\|$ and $|F(x)-(Bx,x)|\leqq K_1\|x\|^2$. It is shown that if K and K_1 are properly restricted by functions of the largest positive characteristic number of B, then Γ has a characteristic vector in S. Further results are also given.

M. M. Day (Urbana, Ill.)

CALCULUS OF VARIATIONS

See also 172, 318.

236:

Glebskiĭ, Yu. V. On the characteristic properties of the solutions of regular and quasiregular problems in the calculus of variations. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 910–912. (Russian)

It is known that a curve Γ which minimizes a regular functional $I(\Gamma)=\int_a^b F(P,\dot{P})dt$ must be smooth [see Busemann and Mayer, Trans. Amer. Math. Soc. **49** (1941), 173–198; MR **2**, 225]. This result is extended to quasiregular problems by modifying the notion of smoothness as follows. A curve Γ in the plane R^2 is termed smooth with respect to the functional $I(\Gamma)$ if the ratio $I(\gamma):F(P_1,P_2-P_1)$ tends to 1 as P_2-P_1 tends to 0, where γ is an arc of Γ from P_1 to P_2. The following theorem concerning the inverse problem is also announced. Let Γ_0 be a curve without self-intersections smooth for the quasi-

regular problem $\int F(P, \dot{P})dt$, where P is restricted to a closed bounded set $D \subset R^2$. Then there exists $G(P) > 0$ such that Γ_0 minimizes $\int G F dt$.

W. H. Fleming (Providence, R.I.)

237:

Fet, A. I. Absolute minimum in a two-dimensional parametric problem on a manifold. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1224–1226. (Russian)

The author sets himself the task of extending Sigalov's existence theorem [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 2(42), 16–101; MR **13**, 257, 1139] to parametric surfaces on a differentiable manifold; a similar program for Plateau's problem was carried out by C. B. Morrey [Ann. of Math. (2) **49** (1948), 807–851; MR **10**, 259]. In the absence not only of proofs, but also to some extent of definitions, the theorems enunciated appear capable of misinterpretation, since they seem to contradict well-known examples such as that of Fleming [Proc. Amer. Math. Soc. **7** (1956), 1063–1074; MR **18**, 489]. A more limited interpretation was almost certainly intended, in view of the limited background of references: otherwise he would presumably have quoted also Sigalov's later work, which deals with the higher topological types but contains an unfortunate oversight [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 53–98; MR **19**, 560]. If the author means to limit himself to the type of the disc his theorems are doubtless correct, but on a manifold this limitation is not very natural.

L. C. Young (Madison, Wis.)

238:

Cinquini, Silvio. Un'osservazione sopra le estremaloidi dei problemi variazionali di ordine n. Boll. Un. Mat. Ital. (3) **12** (1957), 385–393.

For some special variational problems of nth order in ordinary form, a sufficient condition is given that the $(n-1)$th derivative of an extremaloid satisfy a Lipschitz condition.

W. H. Fleming (Providence, R.I.)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 78, 530.

239:

✱Heffter, Lothar. Grundlagen und analytischer Aufbau der projektiven, euklidischen, nichteuklidischen Geometrie. Dritte, wesentlich überarbeitete Auflage. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1958. 191 pp. DM 16.80.

This book has been based on L. Heffter and C. Koehler's Lehrbuch der analytischen Geometrie [2 vols., 1905, 1923, 1929], of which a shortened second edition appeared in 1940 under the title of the present book [MR **2**, 9], now republished in a revised edition by the 95-year-old author. The relationship of four undefined concepts (point, plane, straight line, incidence) is laid down in ten axioms and their dualistic counterpart, creating thus an axiomatic space. To this first set of 20 axioms of incidence is added a second set of 4 axioms of order, but they too can be formulated as axioms of incidence. Now the theorem of Pasch (which is not an axiom here) is deduced from them. The third set of axioms consists of the continuum axiom of Dedekind in projective formulation. A projective theorem, equivalent to the Archimedes axiom, is deduced from it. Using the concept of cross ratio projective geometry or "geometry of incidences" based on these three sets of axioms is developed. Additional axioms lead to parallel and orthogonal geometry. These three geometries and their theorems are now formulated in analytic form. The last chapter gives non-euclidean geometry within this axiomatic framework. The book is the life work of half a century and contains the author's 'final unified formulation' of the different geometries in keeping with Klein's Erlangen program.

S. R. Struik (Cambridge, Mass.)

240:

Szász, Pál. The relation between the Poincaré half-plane and hyperbolic plane-geometry. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 163–184. (Hungarian)

This article is essentially a translation of the German article by the same author [Acta Math. Acad. Sci. Hungar. **4** (1953), 243–250; MR **15**, 645 (first review on the page)].

241:

Makarova, N. M. A correlation rule in the geometry of the triangle. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 135–142. (Russian)

242:

✱Krygowska, Zofia. Konstrukcje geometryczne na płaszczyznie. [Geometrical constructions in the plane.] Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 196 pp. (1 insert) zł. 15.

The book is intended for readers who know about geometric constructions as much as is included in the traditional course in plane geometry of the secondary schools.

After a brief chapter (pp. 6–13) on the history of geometric constructions the author gives a fair account of the power and limitations of the classical constructions with ruler and compasses. She contributes, of course, her share to the sisyphean task of convincing the ever present squarers of the circle and the trisectors of the angle that their misdirected efforts are foredoomed to fail (pp. 119–193).

In the third and last chapter are considered the various other methods of geometric constructions: the use of higher plane curves, the use of the ruler alone, the marked ruler, the Mascheroni constructions, and others. In each case the fundamental theory is given, but applications are few.

The book concludes with a bibliography of more than a dozen items and a detailed table of contents. There is no index.

N. A. Court (Norman, Okla.)

243:

Wunderlich, Walter. Über die Hundekurven mit konstantem Schielwinkel. Monatsh. Math. **61** (1957), 277–311.

The author gives a solution by geometrical methods of the following problem: In an x-y-plane a target Q moves along a straight line with constant velocity v_Q; a pursuer P moves with constant speed v_P in a direction which deviates from QP by the constant "squinting angle" σ; what is the path of P, does P hit the target and, if so, when, where and at what angle? The main tool is the construction of the world line of the pursuer in x-y-t-space where the unit of the time t is so chosen that the world line is inclined at 45° to the x-y-plane. It is shown that the world line lies on a cone or cylinder with the following two properties: (i) it meets the x-y-plane in a conic of eccentricity $\varepsilon = v_Q/v_P$; (ii) its intersection with the plane

at infinity touches the cone $x^2+y^2-t^2=0$ in two points U, V. Four cases are distinguished: (1) $\varepsilon<1$, $|\sigma|<90°$ ("Treffer"): P hits Q; (2) $\varepsilon=1$, $|\sigma|<90°$ ("Nachläufer"): P reaches Q at infinity; (3) $\varepsilon>1$, $|\sigma|<90°$ ("Flüchter"): the distance PQ increases from a certain time onwards; (4) $|\sigma|=90°$ ("Querläufer"): P moves at right angles to PQ and cannot reach Q. These cases are further specified according to whether (a) $|\sin\sigma|<\varepsilon$ (hyperbolic case, U, V real, distinct), (b) $|\sin\sigma|=\varepsilon$ (parabolic case, $U=V$), (c) $|\sin\sigma|>\varepsilon$ (elliptic case, U, V conjugate complex). Each of the eight cases so arising is treated on its own merits.

F. A. Behrend (Melbourne)

244:

Myller, Al.; and Popa, I. Sections dans une prisme triangulaire. Gaz. Mat. Fiz. Ser. A. **2** (1958), 76–84. (Romanian. French and Russian summaries)

Chaque triangle est la projection d'un autre triangle semblable avec un triangle donné.

Du résumé des auteurs

245:

Marzani, Th. Die flächentreue Affinität. Math.-Phys. Semesterber. **5** (1957), 272–280.

Verf. hat einen "Funktionenschieber" konstruiert, welcher die flächentreue Affinität anschaulich macht und gibt einige Anwendungen für den Unterricht an (Krümmung von Kegelschnitten, Teilung einer Hyperbelfläche in flächengleiche Sektionen).

O. Bottema (Delft)

246:

Brauner, H. Über die durch einen quadratischen Komplex der Charakteristik (11) (112) **vermittelte Projektion. I, II.** Monatsh. Math. **62** (1958), 119–145.

Man kann in der konstruktiven Geometrie einen Komplex als Abbildungsmittel verwenden: jedem Raumpunkt wird denn in der Bildebene die Spur des Komplexkegels zugeordnet. Das Komplex heisst zirkular wenn das Bild eines beliebigen Punktes ein Kreis ist. Verf. betrachtet einen zirkularen quadratischen Komplex Σ der Charakteristik (11) (112), der aus ∞^1 Drehnetzen besteht. Er untersucht die Eigenschaften von Σ, insbesondere die Gruppe seiner projektiven Automorphien; es wird im Raum eine quasi-elliptische Metrik eingeführt; Σ ist Tangentenkomplex einer nicht-euklidischen Schraubung und die Abbildung ist der gewöhnlichen Zyklographie analog. Der zweite Teil der Arbeit untersucht die Abbildung des Punkt-, Ebenen- und Geradenraumes.

O. Bottema (Delft)

247:

Bartel, Kazimierz. Geometria wykreślna. [Descriptive geometry.] 4th ed. unaltered. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 427 pp.

The first edition of this book was published some forty years ago, with a second edition in 1922 [Książnica Polska, Lwów]. In the third edition (1948), prepared by Antoni Plamitzer, some minor changes were made, but the book remains basically the same as in the second edition. The present fourth edition is a reprint of the third.

The book is a comprehensive course in descriptive geometry, including what is called engineering drawing and concluding with the theory of shadows (pp. 382–422). The introductory chapter reminds the reader of the basic theorems (and their proofs) of solid geometry that will be made use of in the text. Throughout the book care is taken to present the theoretical foundations on which are based the solutions of the problems considered. The methods follow the classical Mongean lines. Nothing is said about what has in recent years become known as the "direct method".

The book includes more than five hundred and fifty drawings, opens with a very detailed table of contents and ends with a good five page index. There is no bibliography.

N. A. Court (Norman, Okla.)

248:

Jeger, M. Das axonometrische Prinzip im Lichte moderner Begriffsbildungen. Elem. Math. **13** (1958), 1–13.

In his Lehrbuch der darstellenden Geometrie, Birkhäuser, Basel, 1947 [MR **9**, 155], E. Stiefel considers certain generalizations of the axonometric map (general axonometry, pp. 133–135; continuous perspective, pp. 166–168). The present author gives a detailed expository account of these axonometric maps (including a special case of the continuous perspective called geodesic map, whereby the straight lines of space are mapped on a quasigeodesic system of plane curves, i.e., a two-parametric family defined by a differential equation of the type $y''+A(x,y)y'^3+B(x,y)y'^2+C(x,y)y'+D(x,y)=0$). Special emphasis is given to the rôle and various forms of Pohlke's theorem, the significance of the Reidemeister configuration, and the connections with the theory of hexagonal webs.

F. A. Behrend (Melbourne)

249:

Wagner, A. On projective planes transitive on quadrangles. J. London Math. Soc. **33** (1958), 25–33.

A projective plane is said to be transitive on quadrangles (t.q.) if to every ordered pair Q_1, Q_2 of quadrangles of the plane there corresponds at least one collineation of the plane which maps Q_1 upon Q_2. Here it is understood that a quadrangle is an ordered set of four distinct points no three of which are collinear. As is well known, the t.q. planes are precisely those planes with the property that every two coordinate rings are isomorphic. This property has long been known for desarguesian planes and has been established more recently for alternative planes [Bruck and Kleinfeld, Proc. Amer. Math. Soc. **2** (1951), 878–890; MR **13**, 526]. It seems a reasonable conjecture that every t.q. plane is alternative (and hence desarguesian and pappian if finite) but the question remains open. The present results strongly support the conjecture.

The results may be summed up as follows: (A) Every finite t.q. plane is desarguesian. (B) A t.q. plane which possesses a perspectivity (a collineation fixing every point of a line but not every point of the plane) is alternative. (C) A t.q. plane which possesses a (subgroup of the) collineation group which is both transitive and regular on quadrangles is alternative. (D) If a t.q. plane possesses a minimal subplane then every quadrangle generates a minimal subplane which is pappian and has coordinates from a prime field independent of the quadrangle.

Although these results are new, they overlap (as, indeed, the author points out) recent results of Ostrum [Canad. J. Math. **8** (1956), 563–567; MR **19**, 445] and Gleason [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593].

R. H. Bruck (Madison, Wis.)

250:

Deaux, R. Sur quatre homographies binaires. Bul. Inst. Politech. Iaşi **4** (1949), 23–29.

251:

Skopec, Z. A. The mapping of the planes of a three-dimensional Euclidean space onto oriented curves of the third order on a Euclidean plane. Mat. Sb. N.S. **44(86)** (1958), 245–262. (Russian)

The mapping is obtained by means of a third order surface F_3 with a plane of symmetry π, the image plane. The homogeneous coordinates in E_3 are chosen in such a way that the F_3 is represented by $x_3^2x_4=f(x_1, x_2, x_4)$, where f is a homogeneous form of the third order. Then π is represented by $x_3=0$, and there exists a point P $(x_1=0, x_2=0, x_4=0)$ in the improper plane $x_4=0$ which is a flat point of the surface. From P the intersection of a plane σ of E_3 with F_3 is projected into a curve s_3 of π. The intersection of π with F_3 serves as the absolute. Since two planes σ symmetrical with respect to π produce the same s_3 it is necessary to orient the s_3 and make them into "cycles". A cycle has with the absolute three tangencies in points on a straight line and further three common asymptotic directions. Then every plane σ has a one to one image in a cycle s_3. The paper studies the possible cases in detail.

D. J. Struik (Cambridge, Mass.)

252:

Zirilli, Francesco. Su una corrispondenza fra piani determinata da un complesso lineare di piani nello spazio a cinque dimensioni. Rend. Mat. e Appl. (5) **16** (1957), 43–57.

Un complesso lineare di piani in un S_5 associa ad un generico piano α delle S_5 un piano α' sostegno degli S_4 polari, rispetto al complesso, delle rette di α.

La corrispondenza P_α, $\alpha\to\alpha'$ induce una corrispondenza T_v sulla grassmanniana V_9 dei piani dello S_5.

L'A. studia le corrispondenze P_α e T_v, determinando loro proprietà caratteristiche e loro costruzioni geometriche.

Assegna inoltre una elegante costruzione della corrispondenza T tra punti, dello spazio ambiente della V_v, rappresentata dalle stesse equazioni della T_v.

C. Longo (Parma)

253:

Hughes, D. R. Collineations and generalized incidence matrices. Trans. Amer. Math. Soc. **86** (1957), 284–296.

Let a (v, k, λ) configuration π possess a collineation group G of order m. If the points of π fall into w_1 and the lines of π into w_2 transitivity classes, $w_1=w_2=w$. Denote by P_i an element of the ith transitivity class of points, by J_j an element of the jth class of lines, by r_i the order of the subgroup of G fixing P_i, by s_j the order of the subgroup fixing J_j, and by a_{ij} the order of the set of elements x of G such that P_ix is on J_j. If A is the matrix of the a_{ij}, C_1 the diagonal matrix of the S_i^{-1}, C_2 the diagonal matrix of r_i^{-1}, and S the matrix of order w all of whose elements are 1:

$$AC_1A^T=(k-\lambda)C_2^{-1}+\lambda mS,$$
$$A^TC_2A=(k-\lambda)C_1^{-1}+\lambda mS,$$
$$AC_1S=SC_2A=kS.$$

If every r_i and s_j is either 1 or m the group G is called standard. The Hasse-Minkowski theory is applied to the above equations to yield numerical results such as: If π possesses a standard collineation group G of order m fixing N points of π, and if $n=k-\lambda$, $t=(v-n)/m$, $s=(t+n-1)/2$, the equation:

$$x^2+ny^2+(-1)^sm^{n-1}\lambda z^2$$

has a non-trivial integral solution.

Applications are given to possible structures of ternary rings coordinatizing a projective plane and it is noted that standard collineation groups of a plane of order 10 must fix an odd number of points.

J. D. Swift (Los Angeles, Calif.)

254:

Hughes, D. R. Generalized incidence matrices over group algebras. Illinois J. Math. **1** (1957), 545–551.

In the notation of the previous review, let F be a field containing the elements of the matrices C_i and $F(G)$ the group algebra of G over F. If the elements of the matrices A, C_i^{-1}, mS are replaced by sums over the corresponding elements of G in $F(G)$ instead of their enumeration, the resulting equations hold in the matrix algebra over $F(G)$. The original equations then follow by homomorphic mapping of G on the identity group using the rational field. If $\lambda=1$, G of order 2, an isomorphic mapping on the group $\{\pm1\}$ and a theorem of R. Baer on the elements left fixed by a collineation of order 2 yield: If π is a projective plane of order $n\equiv2 \pmod 4$ and if π possesses a collineation of order 2, then $n=2$. Since all known planes have collineations of order 2, this would tend to support the conjecture that there are no planes of singly even order greater than two. The following curious conjecture is advanced by the author: If there exists a projective plane of order 10, it possesses no non-identity collineations. In support of it, considerations of double transitivity are adduced to show that only four types of collineations of prime order can occur in such a plane and the author reports that E. T. Parker has eliminated one of these.

J. D. Swift (Los Angeles, Calif.)

255:

Dembowski, Peter. Verallgemeinerungen von Transitivitätsklassen endlicher projektiver Ebenen. Math. Z. **69** (1958), 59–89.

Let $\mathfrak{E}$ be a (v, k, λ) configuration, and suppose the set of points of $\mathfrak{E}$ is partitioned into classes $\{\mathfrak{P}\}$ and the set of lines of $\mathfrak{E}$ into classes $\{\mathfrak{g}\}$. Suppose $\mathfrak{P}$ and $\mathfrak{g}$ are an arbitrary pair of such classes, and on each line of $\mathfrak{g}$ there are the same number of points of $\mathfrak{P}$ and through each point of $\mathfrak{P}$ there are the same number of lines of $\mathfrak{g}$. Then the partitions $\{\mathfrak{P}\}$ and $\{\mathfrak{g}\}$ define a 'tactical decomposition' of $\mathfrak{E}$. The transitive constituents of a collineation group of $\mathfrak{E}$ always define a tactical decomposition, but the author shows that there are tactical decompositions of some projective planes, for instance (i.e., $\lambda=1$), which cannot arise from any collineation group. He shows first that the number of point classes $\{\mathfrak{P}\}$ is always equal to the number of line classes $\{\mathfrak{g}\}$; this generalizes a theorem due to the reviewer [# 253 above] and to Parker [Proc. Amer. Math. Soc. **8** (1957), 350–351; MR **19**, 163], asserting a similar theorem about transitive constituents with respect to a collineation group. (All three of these proofs appear to have been not only independent, but almost simultaneous.) The author gives some examples of the use of the theorem, which has already proved to have many applications in studying collineations. The second part of the paper considers the incidence structure of the naturally defined object $\mathfrak{E}/\Gamma$, where Γ is a tactical decomposition. It is shown that $\mathfrak{E}/\Gamma$ is a finite projective plane exactly when $\mathfrak{E}$ is a finite projective plane and every class of Γ consists of exactly one object (point or line, as the case may be). Conditions that $\mathfrak{E}/\Gamma$ be a degenerate plane are also studied. The third part is concerned with a sort of "generalized incidence matrix" defined by a tactical decomposition. These matrices satisfy equations which

give rise to a rational congruence, allowing the application of the Hasse-Minkowski theory and the demonstration of the non-existence of certain tactical decompositions; this is analogous to the use of Hasse-Minkowski to show the non-existence of certain (v, k, λ) configurations (i.e., the Bruck-Ryser and Chowla-Ryser theorems). The last part of the paper is very similar to a treatment of collineation groups given by the reviewer [loc. cit.] in which certain (v, k, λ) configurations with certain collineation groups are shown not to exist. *D. R. Hughes* (Chicago, Ill.)

256:

Ostrom, T. G. Dual transitivity in finite projective planes. Proc. Amer. Math. Soc. **9** (1958), 55–56.

It has been shown that a finite projective plane with a collineation group doubly transitive on points is necessarily Desarguesian (the most difficult case, when the order of the plane is a square, has been solved by Ostrom and Wagner in a paper to appear soon). This note considers the following notion: If π is a projective plane and if π has a collineation group transitive on the pairs (p, L), where p is a point, L a line, p not on L, then π is 'dually transitive'. Using the Dembowski-Hughes-Parker theorem on equality of number of transitive constituents of points and lines, the author shows that if π is dually transitive, then it has a collineation group doubly transitive on points. {Reviewer's remarks: The converse is easily proven, and thus we have a simple way to show that double transitivity on points is equivalent to double transitivity on lines.} *D. R. Hughes* (Chicago, Ill.)

257:

Koba, V. I. Some investigations into the geometry of the triangle in the Lobačevskian plane. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 114–128. (Ukrainian)

258:

Makarova, N. M. Trigonometry of the triangle in plane non-Euclidean geometries. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 97–99. (Russian)

259:

Podševkin, Yu. V. Octahedral spaces. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 109–113. (Ukrainian)

CONVEX SETS AND DISTANCE GEOMETRIES

260:

Černikov, S. N. Homogeneous linear inequalities. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 185–192. (Russian)

Proof of results announced in Dokl. Akad. Nauk SSSR (N.S.) **89** (1953), 977–980 [MR **15**, 294]. *T. S. Motzkin* (Los Angeles, Calif.)

261:

Černikova, N. V. Maximum and minimum values of a linear function on a polyhedron. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 193–198. (Russian)

This fundamental optimization problem is here directly solved by comparison and evaluation of subdeterminants. *T. S. Motzkin* (Los Angeles, Calif.)

262:

Isbell, J. R. Homogeneous games. Math. Student **25** (1957), 123–128.

A homogeneous game is a simple game (i.e., each coalition of players either wins all or nothing) such that given any two players, A, B, it is possible to permute the players so that all winning sets go into winning sets and A goes into B. A strong simple game is one in which the complement of every losing coalition is winning. The author obtains results about the existence of strong simple homogeneous n-person games for a class of values of n and of such games with n minimal winning coalitions. He also obtains a bound for n, given the size of a winning coalition in a simple homogeneous game. *H. M. Gurk* (Princeton, N. J.)

263:

***Rabin, Michael O. Effective computability of winning strategies.** Contributions to the theory of games, vol. 3, pp. 147–157. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author's main result is obtained by defining a certain game in terms of an effectively computable function, and showing that if this function enumerates a simple set (in the sense of Post), then there is no effectively computable winning strategy for this game. *R. M. Baer* (Berkeley, Calif.)

264:

***Oxtoby, John C. The Banach-Mazur game and Banach category theorem.** Contributions to the theory of games, vol. 3, pp. 159–163. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author generalizes the Banach-Mazur game of "pick an interval" as follows. Let X be an arbitrary topological space, and $\mathcal{G}$ a fundamental system of neighborhoods (not necessarily open) for X. Let $X=A\cup B$, where A and B are disjoint. Two players (A) and (B) alternately choose sets G_n of $\mathcal{G}$ such that $G_n \supset G_{n+1}$ $(n=1, 2, \cdots)$, with player (A) having first choice. Player (A) wins in case $A\cap\bigcap G_n$ is non-empty. Otherwise (B) wins. Th. 1: (B) has a winning strategy if and only if A is of first category in X. Th. 2: If X is a complete metric space, then (A) has a winning strategy if and only if B is of first category at some point of X. From Th. 1 a new proof of the Banach category theorem is obtained. An example shows that the "necessity" part of Th. 2 is not true if, say, X is any compact Hausdorff space. *W. H. Fleming* (Providence, R.I.)

265:

***Sion, Maurice; and Wolfe, Philip. On a game without a value.** Contributions to the theory of games, vol. 3, pp. 299–306. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

By an example it is shown that Glicksberg's theorem on semicontinuous payoffs [RAND Corp. Report **RM-478**, Santa Monica, Calif., 1950] cannot be extended in certain directions. The example, a modified Blotto game, is somewhat simpler than Ville's original one. By writing each number in the unit interval in its dyadic expansion, this game is converted into a game in extensive form with a countable number of moves, which likewise does not have a value. The latter game can be interpreted as a kind of "game of pursuit". *W. H. Fleming* (Providence, R.I.)

266:
Molnár, J. Über eine Vermutung von G. Hajós. Acta Math. Acad. Sci. Hungar. **8** (1957), 311–314.

The author proves the following Helly-type theorem, conjectured by Hajos. Let S denote a surface not homeomorphic to the 2-sphere, and let $\mathfrak{M}$ be a family of pieces (Elementarflächenstücken) of S. If the product of each 2 pieces is connected, and each three have a common point, then so do all the members of $\mathfrak{M}$.

L. M. Blumenthal (Columbia, Mo.)

267:
Molnár, J. Über eine Übertragung des Hellyschen Satzes in sphärische Räume. Acta Math. Acad. Sci. Hungar. **8** (1957), 315–318.

In a previous note [same Acta **7** (1956), 107–108; MR **18**, 56] the author showed that if any family of simply connected, bounded and closed subsets of the 2-sphere is such that (i) the product of each 2 members of the family is connected, (ii) the product of each 3 is non-empty, and (iii) the 2-sphere is not covered by any four of the sets, then the family has a non-empty product. The present contribution extends this result to bounded, closed, and convex subsets of (1) convex surfaces and (2) the n-sphere, and obtains, among other consequences, a theorem of C. V. Robinson [Amer. J. Math. **64** (1942), 260–272; MR 3, 300] and a spherical analogue of Jung's theorem [J. Reine Angew. Math. **123** (1901), 241–257].

L. M. Blumenthal (Columbia, Mo.)

268:
Hadwiger, H. Einige einfache Sätze über Distanzmittel bei konvexen Körpern. Bul. Inst. Politech. Iaşi **4** (1949), 30–35.

Let $K_1, \cdots, K_r$ be convex bodies in E^n. For any point x define $p_r(x)=\min_{y\in K_r} xy$, $q_r(x)=\max_{y\in K_r} xy$ (xy denotes distance). For $\alpha\geqq 1$ put $M_\alpha(x)=[r^{-1}\sum p_r^\alpha(x)]^{1/\alpha}$, $N_\alpha(x)=[r^{-1}\sum q_r^\alpha(x)]^{1/\alpha}$. The functions $M_\alpha(x)$ and $N_\alpha(x)$ are convex. The sets M, N where $M_\alpha(x)$ or $N_\alpha(x)$ attain their minima are convex. Moreover, $\dim N\leqq 1$, $\dim M\leqq n$ with the inequality for $\alpha>1$.

H. Busemann (Cambridge, Mass.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 266.

269:
Viola, Tullio. Questioni connesse con un problema di teoria degli insiemi sul piano iperbolico o sulla retta proiettiva. Rend. Sem. Mat. Messina **2** (1956), 33–49 (1957).

The author considers sets which the reviewer has named "Sierpiński sets" (that is sets "congruent" to all their maximal proper subsets) on the projective real line, or, equivalently, in the hyperbolic plane. Here congruence is with respect to real linear fractional transformations.

It is shown that, while the Sierpiński sets considered by the author are "usually" everywhere dense, it is possible to construct nowhere dense Sierpiński sets.

E. C. Straus (Los Angeles, Calif.)

270a:
Bagemihl, F. Rectilinear limits of a function defined inside a sphere. Michigan Math. J. **4** (1957), 147–150.

270b:
Piranian, George. Ambiguous points of a function continuous inside a sphere. Michigan Math. J. **4** (1957), 151–152.

270c:
Bagemihl, F. Ambiguous points of a function harmonic inside a sphere. Michigan Math. J. **4** (1957), 153–154.

270d:
Church, P. T. Ambiguous points of a function homeomorphic inside a sphere. Michigan Math. J. **4** (1957), 155–156.

Bagemihl constructs a harmonic function h on the open 3-cell S such that for every point Q of the boundary and every real number r, including the values $\pm\infty$, there is a Jordan arc J_r^Q extending in S from Q such that $h(P)\to r$ as $P\to Q$ along J_r^Q. Church shows that there is an infinitely differentiable autohomeomorphism h of S with the property that for every point Q of the boundary there are two Jordan arcs, J_0 and J_1, extending in S from Q, such that the limits of $h(P)$ as $P\to Q$ on J_0 and as $P\to Q$ on J_1 are different. Other examples constructed in these papers are more or less subsumed in the above.

R. H. Fox (Princeton, N.J.)

271:
Abian, Smbat. A general definition of convergence, continuity, differentiability and integrability. Math. Ann. **134** (1957), 93–94.

The notion of convergence and related concepts are introduced on a set with a structure. A collection $\mathfrak{X}=(X_\iota)_{\iota\in I}$ of subsets of a set X is defined to be a structure on X if $\bigcup_{\iota\in I} X_\iota=X$. A family $\mathfrak{F}$ is defined to be a convergent family if $\mathfrak{F}=(X_\iota)_{\iota\in J}$, $\bigcap_{\iota\in J} X_\iota=L\neq\theta$, and $J=[i\in I \mid X_\iota\cap L\neq\theta]$. $\mathfrak{F}$ is then said to be converging to L. With the aid of this concept of convergence, related concepts such as limit of a subset, convergent subset, limit of a function, continuous, differentiable and integrable functions are defined. It is remarked that the concepts of limit, convergence, continuity, differentiability and integrability in classical analysis and topological spaces are special cases of the corresponding concepts with the same name given in this paper.

Hing Tong (Middletown, Conn.)

272:
Kopřiva, Jiří. Zum Problem der Iteration in der Topologie. Práce Brn. Českoslov. Akad. Věd **29** (1957), 256–276. (Czech. Russian and German summaries)

Let 1 be a (fundamental) set, P1 the system of all the subsets of 1 and u a topology on 1, i.e., a mapping of P1 into P1; one has then the topological space $(1, u)$ as well as the spaces $(1, u^k)$ for every positive integer k, where $u^1X=uX$, $u^{k+1}X=u(u^kX)$. If φ is a mapping of PP1 into P1, i.e., if for every system S of subsets of 1 we have a subset φS of 1, then one defines the corresponding iterations u^η for any ordinal $\eta>0$ in the following way: $u^1X=uX$, $u^{\eta+1}X=u(u^\eta X)$, $u^\lambda X=\varphi S_\lambda$, where S_λ denotes the system of sets u^kX $(k<\lambda)$; λ denotes a limit ordinal. The question arises to study the sequences u^kX (k varying) for every $X\subseteq 1$ and in particular to determine the first ordinals k satisfying $u^{k+1}X=u^kX$ (the case where φ denotes the union was considered by K. Koutský). An ordered pair (α, π) of ordinals is referred to as ordered pair [o.p.] associated to a set $X\subseteq 1$, provided the relations $\eta=\alpha+\pi\tau_\eta+\nu_\eta$, $0\leqq\nu_\eta<\pi$ imply $u^\eta X=u^{\alpha+\nu_\eta}X$. Let MX be the class of all the o.p. associated to X. A period of X is any ordinal $\pi>0$ satisfying $(\alpha, \pi)\in MX$ for some α; let

$\mathfrak{P}X$ be the class of all the periods of X. The beginning of a period π of X is any ordinal α satisfying $(\alpha, \pi) \in MX$; let $A_\pi X$ be the class of all such α's and let $AX = \bigcup_\pi A_\pi X$ ($\pi \in \mathfrak{P}X$). Let $\mathfrak{P}_\alpha X$ denote the class of all the π for which $(\alpha, \pi) \in MX$. Let $\alpha X = \inf AX$, $\pi X = \inf \mathfrak{P}X$. One has $\omega \in \mathfrak{P}X$ (Th. 1). If $\alpha \in AX$, $\pi \in \mathfrak{P}X$, then $(\alpha, \pi) \in MX$ (Th. 10). For ordinals $\alpha \geqq 0$, $\pi > 0$ the relation $(\alpha, \pi) \in MX$ is equivalent with the system $\alpha = \alpha(X) + \pi(X)\lambda$, $\lambda \geqq 0$, $\pi = \pi(X)\mu$, $\mu > 0$ (Th. 11). Let Q be the class of all the $(\alpha X, \pi X)$ with $X \subseteq 1$. The author gives a necessary and sufficient condition for the existence of u and φ such that the equality $B = Q$ holds, B denoting the class of all the ordered pairs (α, π). *Đ. Kurepa* (Zagreb)

273:

Mansfield, M. J. Some generalizations of full normality. Trans. Amer. Math. Soc. **86** (1957), 489–505.

A space X is "m-fully normal" if each (open) covering $\mathscr{A}$ has a refinement $\mathscr{B}$ such that, for each $x \in X$ and each collection of $\leqq \mathfrak{m}$ sets B_λ such that $x \in B_\lambda \in \mathscr{B}$, $\bigcup B_\lambda \subset$ some $A \in \mathscr{A}$. X is "almost-m-fully normal" if $\mathscr{B}$ satisfies the weaker condition: for each $x \in X$ and each set $M\mathrm{CSt}(x, \mathscr{B})$ with $\leqq \mathfrak{m}$ points, $M \subset$some $A \in \mathscr{A}$. (It is not known, however, if every almost-m-fully normal space is m-fully normal.) Almost-2-fully normal Hausdorff spaces are those for which the family of all neighborhoods of the diagonal is a uniformity [H. J. Cohen, C. R. Acad. Sci. Paris **234** (1952) 290–292; MR **13**, 763]. Every linearly ordered space (in the order-topology) is $\aleph_0$-fully normal, and every almost-$\aleph_0$-fully normal space is countably paracompact. [Together, these theorems extend one of B. J. Ball, Proc. Amer. Math. Soc. **5** (1954), 190–192; MR **15**, 976.] The space of ordinals $< \omega_{\alpha+1}$ is $\aleph_\alpha$-fully normal but not almost-$\aleph_{\alpha+1}$-fully normal. Further results include analogues, for these weaker notions, of many of the standard properties of fully normal (=paracompact) spaces. *A. H. Stone* (Manchester)

274:

Kowalsky, Hans-Joachim. Automorphismengruppen topologischer Räume. Math. Nachr. **16** (1957), 309–342.

The author deals with the so-called "problem of Wiener", namely, the relation between a topology and the group of all its automorphisms. But he is apparently unaware of the existence of a literature on the subject, and does not mention in his bibliography either Wiener's original paper, or Colmez's thesis [Portugal. Math. **6** (1947), 119–244; MR **10**, 557] where additional references may be found. With such lack of background, it is inevitable that some of the author's results should duplicate known theorems; this is the case, for instance, for the definition and main properties of what he calls "group topologies", which are identical to the J-topologies of Wiener and Colmez. He has, however, other results which seem more original, in particular the fact that, with the exception of a rather uninteresting type, any topology τ is the supremum of the "group topologies" constructed for all the groups of permutations having the property that a permutation of such a group which leaves invariant the points of a set in a filter converging (for τ) to a point p, also leaves p invariant. This leads him to the investigation of the relation between systems of groups of permutations and the topologies obtained by taking suprema of the "group topologies" of the system, in particular the relation between the initial system of groups and the group of automorphisms of the corresponding topology. Finally he shows that, for a large class of topologies, the group structure of their groups of automorphisms is not arbitrary: in particular, their center must reduce to the identity. As special examples are considered the symmetric group on a set E acting on the space of ultrafilters on E, and the projective linear group acting on the projective space; their "group topologies" are determined and studied.

J. Dieudonné (Evanston, Ill.)

275:

Gál, István S. On the continuity and limiting values of functions. Trans. Amer. Math. Soc. **86** (1957), 321–334.

For a real-valued function f on a real interval I, the following two results go back to W. H. Young [Quart. J. Math. Oxford Ser. **39** (1908), 67–83]. (In what follows, "nearly everywhere" means "except on a countable set".) (1) The right and left upper limits of f coincide nearly everywhere in I. (2) For nearly every a in I, $f(x)$ clusters at $f(a)$ as $x \to a$. As a corollary of (1) and (2), one has that (3) if the right and left limits of f exist nearly everywhere in I, then f is continuous nearly everywhere in I. The author establishes, in a quite abstract and general setting, some theorems on limits and continuity which include generalizations or improvements (some new and some old) both for the above results of Young and also for a theorem of H. Blumberg [Fund. Math. **16** (1930), 17–24] on arbitrary functions of two variables, with applications to generalized derivatives.

The result described below is typical of the author's general theorems. Let $f: X \to Y$ be a function. Let there be an uncountable subset A of X on which is defined a topology $\mathscr{T}$ such that every uncountable subset A^* of A has an accumulation point in A^*. With each α in A let there be associated two filters $\mathscr{F}_\alpha^1$ and $\mathscr{F}_\alpha^2$ in X such that $\alpha \in F_\alpha^2$ for every set F_α^2 belonging to $\mathscr{F}_\alpha^2$. Suppose further that for every α in A and every F_α^1 in $\mathscr{F}_\alpha^1$ there is a neighborhood N of α with respect to $\mathscr{T}$ such that every $\mathscr{F}_\beta^2$ for which $\beta \in N$ and $\beta \neq \alpha$ has a member F_β^2 which is a subset of F_α^1. Finally, suppose there is given on Y a uniform structure which satisfies the first axiom of countability. Where $x \in A$ and i is 1 or 2, we say f is continuous at x with respect to $\mathscr{F}_x^i$ provided for every neighborhood U of $f(x)$ there is a set F belonging to $\mathscr{F}_x^i$ such that $f(F) \subset U$. The author's conclusion from all these hypotheses is that the set of all points x in A at which f is continuous with respect to $\mathscr{F}_x^1$ but discontinuous with respect to $\mathscr{F}_x^2$ is countable.

T. A. Botts (Charlottesville, Va.)

276:

Smirnov, Yu. An instance of a one-dimensional normal space contained in no one-dimensional bicompact space. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 939–942. (Russian)

The dimension in question is ind; for dim and Ind, every n-dimensional normal space is contained in an n-dimensional bicompact space. The construction utilizes Lunc's construction [same Dokl. **66** (1949), 801–803; MR **11**, 46] of a bicompact space X with ind $X = 2$, dim $X = 1$.

J. Isbell (Seattle, Wash.)

277:

Jones, F. Burton. Moore spaces and uniform spaces. Proc. Amer. Math. Soc. **9** (1958), 483–486.

The author gives an example of a connected, locally connected, complete Moore space which fails to be completely regular at one point and therefore is not a uniform

space. He first constructs, by topologizing a half-plane, a non-normal uniform Moore space Λ. The desired example is a union of countably many duplicates of Λ plus an extra point. *M. E. Shanks* (W. Lafayette, Ind.)

278:

Jones, F. Burton. R. L. Moore's axiom 1′ and metrization. Proc. Amer. Math. Soc. **9** (1958), 487.

Theorem: Let $G_1, G_2, \cdots$ be a sequence of open coverings of the Hausdorff space S such that (1) $G_{n+1} \subset G_n$, and (2) H and K closed disjoint with H compact implies that there is an n such that no element of G_n meets both H and K. Then the space S is metrizable.

M. E. Shanks (W. Lafayette, Ind.)

279:

Kodama, Yukihiro. On homotopically stable points and product spaces. Fund. Math. **44** (1957), 171–185.

A point $x_0 \in X$ is homotopically n-labile (n-HL) if for each neighborhood $U \supset x_0$ there is a neighborhood V, $x_0 \in V \subset U$, such that any $f: S^n \to V - x_0$ is nullhomotopic in $U - x_0$; x_0 is HL^n if it is i-HL for $0 \leq i \leq n$. x_0 is HL if there is a neighborhood U and a deformation of X rel $X-U$, keeping the image of U in U, and uncovering x_0. It is shown that if X is an ANR for metric spaces (ANRM) and $\dim X \leq m < \infty$, x_0 is HL if and only if it is HL^{m-1}. Main theorem: If X_i, $i=1, 2$, are non-degenerate arc-connected locally compact finite-dimensional ANRM, $(x_1, x_2) \in X_1 \times X_2$ is not HL if and only if the local integral Čech homology group $H_n((x_1, x_2)) \neq 0$ for some $n \geq 0$; for the space X_1 alone, $x_1 \in X_1$ is HL if and only if it is HL^1 and $H_n(x_1) = 0$ for all $n \geq 0$. In cartesian products, it is shown that $\dim(X_1 \times X_2) = \dim X_1 + \dim X_2$ under either of the two conditions: (1) X_1 locally compact fully normal, X_2 locally compact 2-dimensional ANRM; or (2) X_i, $i=1, 2$, locally compact ANRM, $\dim X_i \leq m_i < \infty$, and each X_i has a point x_i which is HL^{m_i-2} but not (m_i-1)-HL. *J. Dugundji* (Los Angeles, Calif.)

280:

Whyburn, Gordon T. Uniform convergence for monotone mappings. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 992–998.

Verfasser beschreibt eine Methode zum Nachweis gleichmäßiger Konvergenz, die die Aufstellung recht allgemeiner Kriterien gestattet. — Die Räume X und Y seien Kontinua (metrisch, separabel, kompakt, zusammenhängend). Y heißt semilokalzusammenhängend (semilocally connected), wenn es um jeden Punkt beliebig kleine (offene) Umgebungen gibt, deren Komplement nur in endlich viele Komponenten zerfällt. Eine Abbildung $f(X) = Y$ heißt monoton, wenn $f^{-1}(y)$ für jedes y ein Kontinuum ist. Dann lautet ein typischer Satz der Arbeit: Voraussetzung: Die f_n sind monotone Abbildungen von X auf den semilokalzusammenhängenden Raum Y. Es gibt ein f mit den Eigenschaften: (1) bei beliebigem y trifft $f(X)$ jede Komponente von $Y-y$; (2) um jedes x gibt es beliebig kleine Umgebungen, auf deren Rand C gilt $\limsup[f_n(C)] \subset f(C)$. Behauptung: Die f_n konvergieren gleichmäßig in X gegen f. Ist Y lokalzusammenhängend, so ist f monoton. — Verfasser behandelt außerdem sowohl speziellere Fälle (Y zyklisch; X eine Kurve) als auch allgemeinere (nichtkompakte Räume; quasi-offene statt monotone f_n). Ferner gibt er Anwendungen auf reelle Funktionen und auf Funktionen, deren Definitionsbereich ein Kreis ist. *K. Zeller* (Tübingen)

ALGEBRAIC TOPOLOGY

See also 46, 317.

281:

Puppe, Dieter. Homotopie und Homologie in abelschen Gruppen- und Monoidkomplexen. I, II. Math. Z. **68** (1958), 367–406, 407–421.

The paper begins with an account of complete semi-simplicial (CSS) complexes and groups; for the latter homotopy groups can be defined (following J. C. Moore) by forming "homology"-groups using, not the free group generated by the complex, but the given group-structure.

Using this method, the author (still following J. C. Moore) obtains homomorphisms

$$\pi_q(K) \xrightarrow{h_*} H_q(K, E) \xrightarrow{g_*} \pi_q(K)$$

such that $g_* h_* =$ identity, where K is any abelian CSS group, E the subcomplex generated by the unit, and h_* is the Hurewicz-homomorphism.

There follows the first published exposition of the realisation of a CSS-complex K by a CW-complex $|K|$, originally due to Milnor; it is proved that the realisation of a CSS group K is a topological group if K is countable; otherwise the groupoperations are at least continuous on compact subsets of $|K| \times |K|$. The homology-groups of a CSS complex and its realisation are isomorphic, as are the corresponding homotopy-groups, at least for CSS-groups. {The result is actually true in general.}

These tools are applied to a recent result of Dold and Thom on the homotopy-groups of infinite symmetric products.

Let K be a CSS complex and L a subcomplex; by ΓK denote the free group generated by K. Then

$$\pi_q(|\Gamma K/\Gamma L|) = \pi_q(\Gamma K/\Gamma L)$$

by the realisation-theorem just quoted;

$$\pi_q(\Gamma K/\Gamma L) = H_q(|K|, |L|)$$

by the way the homotopy groups have been defined for CSS groups.

Now, let L be the subcomplex generated by a base-point only and write $\bar{\Gamma} K = \Gamma K/\Gamma L$. Then $\bar{\Gamma} K$ is an abelian group containing a generating submonoid SPK, the infinite symmetric product of K. The result of Thom and Dold now follows from the fact that $|SPK| = SP|K|$ if K is countable, and the following proposition. Let K be a CSS-group and LCK a submonoid containing the unit in each dimension, generating K and such that $|L|$ is connected. Then the injection $|L| \to |K|$ induces isomorphisms in all homotopy-groups. Namely, putting all these facts together, we obtain

$$\pi_q(SP|K|, 0) = H_q(|K|, e),$$

e denoting a base-point and 0 the (corresponding) zero of $SP|K|$. *V. Gugenheim* (Baltimore, Md.)

282:

Postnikov, M. M. Homology theory of smooth manifolds and its generalizations. Advancement in Math. **3** (1957), 503–546. (Chinese).

A translation of the Russian article in Uspehi Math. Nauk (N.S.) **11** (1956), no. 1(67), 115–166 [MR 18, 409].

283:

Massey, W. S. On the cohomology ring of a sphere bundle. J. Math. Mech. **7** (1958), 265–289.

The first part of the paper deals with the multiplicative structure of the cohomology ring (with integers Z, rational numbers, or integers mod m as coefficients) of a sphere space E (locally trivial fibre space whose fibre is a S^{k-1}). The author assumes that the local system of groups defined by $H^{k-1}(S_x^{k-1})$ is a simple system, S_x^{k-1} denoting the fibre over the point x of the base space B.

Denoting by

$$\cdots \xrightarrow{\psi} H^{q-k}(B) \xrightarrow{\mu} H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\phi} H^{q-k+1}(B) \xrightarrow{\mu} \cdots$$

the Gysin sequence of the sphere space, where π is the projection and μ multiplication by the characteristic class $W_k \in H^k(B; Z)$, the author computes the $z \in H^*(B)$ such that $\pi^*(z)=(\pi^*u)(\psi^{-1}v)$ for $u, v \in H^*(B)$, $uv=0$ and $\mu(v)=0$. A second theorem, due to the reviewer [C. R. Acad. Sci. Paris **241** (1955), 1021–1023; MR **17**, 993], expresses $\psi(xy)$ in terms of $u=\psi(x)$, $v=\psi(y)$, where $u, v \in H^*(B)$ and $\mu(u)=\mu(v)=0$. In the first case, $z=\pm\langle u, v, W_k\rangle$, in the second case $\psi(xy)=\pm\langle u, W_k, v\rangle$ (triple products, signs depending on the parity of the degree). The proof uses R. Thom's expression of the Gysin sequence in terms of the cohomology sequence of the mapping cylinder of the projection [Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 109–182; MR **14**, 1004] and the formula $(ab)\cup_1 c=a(b\cup_1 c)+(a\cup_1 c)b$ due to the reviewer [C. R. Acad. Sci. Paris **241** (1955), 923–925; MR **17**, 396].

If $W_k=0$, the cohomology ring of E is a quadratic extension of the cohomology ring of B, and can be described by two classes β (of dimension k) and α (of dimension $2k$); β is the $(k-1)$-dimensional Stiefel-Whitney class W_{k-1} for k even, or W_{k-1} reduced mod 2 when k is odd (the proof uses again Thom's mapping cylinder); $4\alpha+\beta^2$ is an invariant of the sphere space E; when E is a sphere bundle with $SO(k)$ as structure group and B is a polyhedron, $4\alpha+\beta^2=P_{2k-2}$ (Pontrjagin class of dimension $2k-2$). The theorem is proved for a universal bundle (for sphere bundles with vanishing characteristic class W_k) by considering $B_{SO(k-1)}$ as a $(k-1)$-sphere bundle over $B_{SO(k)}$, where $B_{SO(k)}$ is the classifying space of the universal principal $SO(k)$-bundle. The proof uses results of A. Borel [mimeographed lecture notes, Department of Mathematics, University of Chicago, 1954; Bull. Amer. Math. Soc. **61** (1955), 397–432; MR **17**, 282] on $H^*(B_{SO(k)})$. For n odd, the Pontrjagin class W_{k-1} of $B_{SO(k)}$ is mapped onto the square of the Euler-Poincaré class W_{k-1} of $B_{SO(k-1)}$. — As, for an orientable 2-sphere bundle with vanishing characteristic class W_3, the set of all α is the secondary obstruction (on the 4-skeleton of B), the necessary and sufficient condition (if $H^4(B; Z)$ has no 2-torsion) for the existence of a cross-section over the 4-skeleton is the existence of $\beta \in H^2(B; Z)$ such that $\beta \equiv W_2 \bmod 2$ and $\beta^2=P_4$.

From the knowledge of certain Pontrjagin classes for complex or quaternionic projective spaces, it follows that the $2m$-dimensional complex projective space (of $4m$ real dimensions), for $m>1$, and the m-dimensional quaternionic projective space, for $m>2$, cannot be imbedded differentiably in the euclidean R^{6m+1}.

G. Hirsch (Brussels)

284:

Podševkin, Yu. V. An application of a method of generalized mapping of the torus to obtain a three-dimensional manifold. Kiiv. Derz. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 103–108. (Ukrainian)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 179.

285:

Šingurov, O. P. On numerical formulas for curvature and torsion of a curve. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 117–126. (Russian)

286:

Koroleva, M. S. Extremal properties of Δ-curves. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 127–134. (Russian)

287:

Leonova, V. V.; and Šingurov, O. P. Classification of the points of a parametrized curve in the Euclidean plane. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 101–115. (Russian)

288:

Sun'hè-šèn [Sun, Che-shen]. Some criteria of rigidity for surfaces of revolution. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 758–761. (Russian)

The author's starting point is the theorem of I. N. Vekua [Czechoslovak Math. J. **6(81)** (1956), 143–160; MR **19**, 308] that an ovaloid with one plane opening, in ideal contact with an orthogonal, ideally smooth and absolutely rigid surface, is rigid with respect to one infinitesimally small bending. In the present paper some criteria of rigidity are given in the non-orthogonal case for an ovaloid of rotation with one plane opening, which can also serve for certain surfaces of rotation with positive curvature. If S is a surface of revolution with axis along the z-axis and is given by $\bar{r}(z, \theta)=z\bar{k}+\rho(z)\bar{e}(\theta)$, $\bar{k}\cdot\bar{e}=0$, then the bending vector can be given the form $\bar{U}(z, \theta)=u\bar{e}+v\bar{g}+w\bar{k}$, where $\bar{g}(\theta)=\bar{k}\times\bar{e}$, and u, v, and w are functions of z and θ. From the condition $d\bar{r}\cdot d\bar{U}=0$ follow the equations

$$\rho'(z)u_z+w_z=0,\ v_\theta+u=0,\ \rho(z)v_z+\rho'(z)(u_\theta-v)+w_\theta=0.$$

Then the main theorem proved is that for a contour connection along the contour L of S (of positive curvature) of the form $w-u\cot\theta=0$ there will be rigidity if $0\leqq\theta\leqq\theta_0$, $\theta_1<\theta<\pi$, and that there exists an infinite sequence of angles θ_k, $k\geqq 2$, $\theta_0<\theta_k<\theta_1$ tending toward θ_0 for which the system of three differential equations will have a non-trivial solution, hence S will not be rigid. Examples discussed are given by $\rho=C\sqrt{(R^2-(z-R)^2)}$, $0\leqq z\leqq z_1$, $C>0$, $R>0$ [cf. E. Rembs, Math. Ann. **111** (1935), 587–595], and $\rho=Cz^\alpha$, $0\leqq z\leqq z_1$, $0<\alpha<1$, $C>0$.

D. J. Struik (Cambridge, Mass.)

289:

Şaichin, A. Sur les asymptotes d'une courbe plane. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 11, 25–27. (Romanian. Russian and French summaries)

On montre que les deux définitions habituelles de l'asymptote ne sont pas équivalentes, l'une étant plus générale, et on étudie un cas d'équivalence.

Résumé de l'auteur

290:

Calapso, Maria Teresa. Sulle curve a flessione costante. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 438–442.

Si enuncia il seguente elegante risultato che verrà dimostrato in un successivo lavoro.

Considerate due qualsiasi funzioni $f(t)$ e $\varphi(t)$ soddisfacenti all'equazione

(a) $$f'''=\frac{ic\varphi'''}{\sqrt{(\varphi'^2-2\varphi\varphi'')}} \quad (i=\sqrt{-1};\ \varphi'''\neq 0)$$

e le curve isotrope descritte rispettivamente dai punti X_f (di coordinate $x=tf''-f$, $y=\frac{1}{2}(1-t^2)f''+tf'-f$, $z=i(\frac{1}{2}(1+t^2)f''-tf'+f)$) e X_φ, la più generale curva gobba a flessione costante c può venire rappresentata dalle equazioni parametriche, $X=X_f-(f'''/\varphi''')X_\varphi$, e la torsione, dipendente soltanto dalla φ, è data da $1/T=c\varphi^2\varphi''/(\varphi'^2-2\varphi\varphi'')^{3/2}$.

Integrando la (a), data la φ, segue il notevole risultato: le curve a flessione costante possono tutte venire determinate con una sola quadratura.

Si ritrovano vari noti esempi di tali curve e si estendono i risultati al problema della determinazione delle curve aventi il raggio di flessione funzione lineare dell'arco.

C. Longo (Parma)

291:

Mirguet, Jean. Sur la convergence biunivoque des plans tangents à une orthosurface. C. R. Acad. Sci. Paris **246** (1958), 1956–1957.

En théorie des surfaces, on établit, dans les conditions de dérivabilité traditionnelles, que le long d'une ligne asymptotique, le plan tangent pivote autour de la tangente; j'examine si la propriété subsiste pour des orthosurfaces plus générales et propose un exemple affirmatif déduit de la convergence biunivoque des plans tangents.

Résumé de l'auteur

292:

Springer, C. E. Union parallel displacement. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 104–109.

A curve on a surface is called a union curve (relative to a given rectilinear congruence) if its osculating plane at each point contains the line of the congruence which passes through the point. Continuing his study of union curves [see Canad. J. Math. **2** (1950), 457–460; MR **12**, 358], the author now defines union parallel displacement analogously to Levi-Civita parallelism, replacing the usual coefficients of connection (Christoffel symbols) by new ones which occur in the differential equations defining union curves. It is shown that a congruence may be chosen so that if a surface vector λ^α is displaced around an infinitesimal circuit on the surfaces by union parallelism with respect to the congruence, the vector λ^α returns to its original position.

A. Fialkow (Brooklyn, N.Y.)

293:

Santaló, L. A. Some properties of the local conformal representation of one surface on another. Rev. Un. Mat. Argentina **18** (1957), 45–52. (Spanish. English summary)

Let P, P' be two corresponding points in a conformal correspondence (local) between two twice differentiable surfaces S, S' and let π, π' be the tangent planes at P, P' respectively. Let C be a variable curve on S through P and $Q\in\pi$ be its center of geodesic curvature. If C' is the transformed curve of C and $Q'\in\pi'$ is its center of geodesic curvature at P', then we prove the following theorem: the correspondence $Q\to Q'$ between π and π' is a projectivity.

As immediate consequences we obtain, among others, the following properties: a) By a conformal correspondence between two surfaces S, S', the geodesic curves through a point P of S transform in curves whose centres of geodesic curvature at P' lie on a line; b) if more than a geodesic curve through P transform in a curve of null geodesic curvature at P', all geodesic curves through P transform in curves with the same property; c) the centers of geodesic curvature of all loxodromes of a surface of revolution through a point P lie on a line. When S' is a plane these theorems may have applications to cartography. (Author's summary.)

C. Longo (Parma)

294:

Marcus, F. Sur les surfaces minima qui sont en même temps minima projectives et minima affines. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 331–334.

Cet article est consacré à l'étude des surfaces minima qui jouissent des deux propriétés indiquées dans le titre. Ces surfaces sont caractérisées dans la famille des surfaces minima par la possibilité de mettre leur élément linéaire sous la forme $ds^2=(Au+Bv+C)^2(du^2+dv^2)$, u, v étant les paramètres asymptotiques, et A, B étant liés par la relation $A^2+B^2+1=0$. Elles constituent une classe de surfaces imaginaires à ajouter aux classes des surfaces minima affines déterminées par Thomsen. L'auteur donne leurs équations finies; il en indique les principales propriétés géométriques, et fait connaître un rapprochement intéressant entre ces surfaces et les congruences I des normales aux surfaces minima admettant pour nappes de leurs développées deux surfaces elles-mêmes minima: Les surfaces moyennes des congruences I et les surfaces minima envisagées s'assemblent par couples de surfaces conjuguées en applicabilité au sens d'O. Bonnet. Les congruences I sont des congruences W particulières à surfaces focales minima; les lignes de courbure ne se correspondent pas sur les surfaces focales, et cette circonstance fournit à l'auteur l'occasion de reconnaître que, contrairement à une affirmation de L. Bianchi, il existe d'autres congruences W, à surfaces focales minima, que les congruences W (de Thybaut) sur les nappes focales desquelles se correspondent les lignes de courbure.

P. Vincensini (Marseille)

295:

Kovancov, N. I. Ruled manifolds in a complex of lines. Zaporiz. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **2** (1956), 97–123. (Ukrainian)

This is a report on the Cartan-Finikov method of applying the theory of exterior differential forms to line complexes. It is illustrated by examples; for instance, there is a discussion of normal and parabolic congruences. The Russian papers quoted are by A. M. Vasil'ev [Dokl. Akad. Nauk SSSR (N.S.) **61** (1948), 189–191; MR **10**, 64], M. A. Akivis [ibid. **61** (1948), 181–184; **65** (1949), 429–432; MR **10**, 400; **11**, 134], and V. I. Korovin [ibid **72** (1950), 837–840; MR **12**, 281].

D. J. Struik (Cambridge, Mass.)

296:

Kovantsov, N. I. A line geometry analogue to a triply-orthogonal system of surfaces. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 497–500. (Russian)

A correspondence is established between systems of triply orthogonal surfaces and line complexes consisting of certain sets of ∞^1 W-congruences, with the aid of the analytical apparatus established in his previous paper [Ukrain. Mat. Ž. **8** (1956), 140–158; MR **18**, 505]. In this correspondence a W-congruence corresponds to a surface S of a triply orthogonal system, a ruled surface of the congruence to a curve on S, principal surfaces of the complex to lines of curvature on S, quadrics in the complex to circles, and a developable surface of a congruence to isotropic curves on S. Special cases considered are

triply orthogonal systems of which one family of Lamé is orthogonal to a congruence of lines or circles, and those of which one such family consists of spheres or planes. Complexes which split up into ∞^1 such W-congruences have already been studied by A. M. Vasil'ev, [Dokl. Akad. Nauk. SSSR (N.S.) **61** (1948), 189–191; MR **10**, 64].

D. J. Struik (Cambridge, Mass.)

297:

Yaglom, I. M. Elementary theory of Laguerre transformations. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 3–48. (Russian)

298:

Fava, Franco. Sul comportamento di elementi curvilinei assiali in relazione a trasformazioni puntuali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 60–70.

Su una superficie S "elemento curvilineo assiale", rispetto ad una congruenza K di rette avente S come direttrice, è un elemento differenziale di una curva Γ (di S) avente in ogni suo punto P piano osculatore contenente la retta di K passante per P.

Considerate due superficie S, $\bar{S}$ (non sviluppabili), due congruenze di rette K, $\bar{K}$ aventi rispettivamente S ed $\bar{S}$ come direttrici ed una corrispondenza puntuale T tra le due superficie, l'Autore dimostra che per ciascuna coppia di punti corrispondenti si hanno tre direzioni caratteristiche, tali che ai tre E_2 assiali da esse determinati corrispondono $\bar{E}_2$ anche essi assiali.

Si determinano poi le coppie di congruenze K, $\bar{K}$ che risultano associate ad un campo di direzioni caratteristiche assegnato su ognuna delle due superficie.

C. Longo (Parma)

299:

Salini, Ugo. Enti affini legati al generico punto di una curva "s" del piano affine. Atti Accad. Peloritana Pericolanti. Cl. Sci. Fis. Mat. Nat. (3) **3(48)** (1945–49), 103–135 (1953).

In un generico punto A di una curva Γ di un piano affine è determinata la curva Γ^* luogo del punto medio dei due punti in cui le parallele alla tangente in A a Γ intersecano Γ nell'intorno di A. Curva "s" è una curva Γ tale che il luogo dei centri del fascio di coniche iperosculatrici a Γ^* in A è la tangente in A a Γ. L'Autore caratterizza coniche del detto fascio e dell'analogo fascio relativo a Γ in relazione ad enti affini di Γ.

C. Longo (Parma)

300:

Čašečnikov, S. M. The theory of the field of local hypercones in X_n. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 765–768. (Russian)

Central hypercones in a central affine E_n are determined by their directrix, a regular hypersurface X_{n-2}, given by $x^a=l^a(\eta^a)$, $a, b, \cdots=1, \cdots, n-2$. When in the X_{n-2} a certain scalar density $\mathfrak{M}$ of weight $2/(n-2)$ is covariant constant, the directrix is called normalized. Then for the contracted object of the affine connection in X_{n-2}, $G_a=\partial_a \ln \mathfrak{M}$. A field of normalized directrices $x^a=l^a(\xi^\beta, \eta^\varepsilon)$ leads to a field of local hypercones. The x^a are given but for transformations of the form $x^a/\sigma(\xi^\beta)$. Three W-densities $\mathfrak{n}_a$, m, $\mathfrak{n}$ of weight $-2/(n-2)$, 0, $-2/(n-2)$, depending on ξ^α, η^ε, are introduced: $\mathfrak{n}_a=\mathfrak{n}_{a\alpha}d\xi^\alpha$, $m=m_\alpha d\xi^\alpha$, $\mathfrak{n}=\mathfrak{n}_\alpha d\xi^\alpha$; and they satisfy the equations

$$\overset{2}{\nabla}_b \mathfrak{n}_a=-\mathfrak{G}_{ba}m+v_{ba}\mathfrak{n};\quad \overset{2}{\nabla}_a m=\mathfrak{H}^b_{\cdot a}\mathfrak{n}_b+\mathfrak{W}_a\mathfrak{n};\quad \overset{2}{\nabla}_a \mathfrak{n}=\mathfrak{n}_a,$$

where $\mathfrak{H}^b_{\cdot a}=\mathfrak{G}^{bc}h_{ca}$, $\overset{2}{\nabla}_a$ the covariant operator with respect to the connection $\overset{2}{G}{}^c_{ba}=G^c_{ba}+\mathfrak{G}^{cd}\mathfrak{A}_{bad}$, the $\mathfrak{G}$ and $\mathfrak{A}$ are densities of weight $-2/(n-2)$, on which v_{ba} and $\mathfrak{W}_a$ depend. On this notation see further the papers by V. V. Vagner [Trudy Sem. Vektor. Tenzor. Analizu **6** (1948), 257–364; **8** (1950), 11–72; MR **14**, 1124; **13**, 281]. The resulting linear connection with $\delta\eta^a=d\eta^a+\Gamma^a=0$ is based on the object Γ^a as a linear combination of $\mathfrak{n}_b$, m and $\mathfrak{n}$, with corresponding identities in exterior differential forms. This leads to the theorem that a field of normalized X_{n-2} in X_n can for $n\geqq 4$ be transformed (with the aid of local similarity transformations) into a constant field if and only if $[\partial\Gamma^a]-[\Gamma^e\partial_e\Gamma^a]=0$, $D\mathfrak{G}_{ba}=0$, $D\mathfrak{A}_{cba}=0$, and, for $n=4$, $Dh_{ba}=0$; D being defined in a certain way as an invariant differential operator. The same condition holds for hypercones in X_r of X_n.

D. J. Struik (Cambridge, Mass.)

301:

Homann, Frederick A. On transformations preserving Laguerre-Forsyth canonical form. Proc. Amer. Math. Soc. **9** (1958), 408–411; addendum, **10**, 174..

The author considers a linear differential equation in the Laguerre-Forsyth canonical form:

$$x^{(n)}+C_{n,3}p_3(t)x^{(n-3)}+C_{n,4}p_4(t)x^{n-4}+\cdots+C_{n,n}p_n(t)x=0,$$

where $p_i(t)$ are regular functions in $-\infty<t<\infty$, and $C_{n,i}$ are binomial coefficients. The set of n linearly independent solutions $x_1(t), \cdots, x_n(t)$ defines a curve, C, in projective space S_{n-1}. The linear osculating spaces of this curve are $S_k(t)$, $0\leqq k\leqq n-1$. The set of kth derivatives of the above solutions (for fixed t) give a "kth derivative point," and the set of the first $k+1$ derivative points is a basis for S_k at the fixed value of t.

The equation above is preserved by the group T of transformations $\bar{t}=(\alpha t+\beta)/(\gamma t+\delta)$, $\bar{x}=\lambda(d\bar{t}/dt)^{(n-1)/2}x$, but the derivative points move about in S_k. The purpose of the paper is to study the locus C_k of the kth derivative point in S_k under the transformations of T. Equations for C_k are obtained, and an intrinsic interpretation given.

The author observes in the addendum that his results are essentially contained in a paper of Bompiani, Ann. Sci. Univ. Jassy **23** (1937), 25–105.

C. B. Allendoerfer (Seattle, Wash.)

302:

Karapetyan, S. E. The second order Lie surface for ruled surfaces of a congruence. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 177–179. (Russian)

The lines of a linear complex (in projective three-space) can be arranged on ruled surfaces. Every generating line of such a surface, together with two neighboring lines, determines the osculating hyperboloid along this line, or quadric of Lie. With the aid of Finikov's ω-theory and the representation of the lines in five-dimensional projective theory an invariant equation of this quadric is obtained, invariant with respect to all measuring tetrahedra of the first order. A number of geometrical applications follow: e.g., the theorem that every half-quadric of Lie (set of ∞^1 rulings of the same kind) connected with a line of the congruence, belongs to one linear complex if and only if the congruence is a W-congruence.

D. J. Struik (Cambridge, Mass.)

303:

Marcus, F. Une nouvelle caractérisation des réseaux et surfaces $\mathscr{E}$ de Cartan. Czechoslovak Math. J. **7(82)** (1957), 308–313. (Russian summary)

E. Cartan a démontré l'existence d'une classe de surfaces douées d'un réseau conjugué tel que $h=\bar{h}$, $k=\bar{k}$; h, k,

h, $\bar{h}$ étant les invariants des équations ponctuelles et tangentielles de Laplace correspondantes. Ces réseaux et les surfaces respectives sont nommés, par E. Cartan, réseaux et surfaces $\mathscr{E}$. Dans l'article présent l'auteur caractérise les réseaux et surfaces en question de la manière suivante: Si les lignes d'un réseau conjugué sont des pangéodésiques, le réseau et la surface appartiennent à la classe $\mathscr{E}$. *O. Borůvka* (Brno)

304:

Tutaev, L. K. On the theory of surfaces in Lorentz space of three dimensions. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 59–64. (Russian)

305:

Matsumoto, Makoto. The differential geometry of spaces with analytic distances. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 119–141.

The author considers an n-dimensional manifold S of class C^ω with a distance function $d(p, q)$ defined for all points $p, q \in S$ and assumes that

$$g(x, y)=-\tfrac{1}{2}[d(p, q)]^2$$

is a function of class C^ω of coordinates (x^i) of p and (y^i) of q. $g(x, y)$ is called the fundamental function of the space S.

If we put

$$g_i=\partial g/\partial x^i, \quad g_{(i)}=\partial g/\partial y^i,$$

then g_i are components of a covariant vector under the coordinate transformation $\bar{x}^i=\bar{x}^i(x)$ and $g_{(i)}$ are those of a covariant vector under the coordinate transformation $\bar{y}^i=\bar{y}^i(y)$. We may consider a tensor of general type, for example $T^{i}{}_{j(k)(l)}$, whose transformation law under the coordinate transformations $\bar{x}^i=\bar{x}^i(x)$ and $\bar{y}^i=\bar{y}^i(y)$ is obvious. g_i and $g_{(i)}$ are called slope vectors. $g_{i(j)}=\partial^2 g/\partial x^i \partial y^j$ is called the relative metric tensor and the determinant $|g_{i(j)}|$ is supposed to be different from zero in the domain under consideration. $g_{i(j)}$ and its inverse $g^{i(j)}$ are used to raise and lower the indices i or (j) as usual. The contraction with slope vectors is denoted by 0, for example, $T^0=g_i T^i$, and $T_{(0)}=g^{(i)}T_{(i)}=g^{j(i)}g_j T_{(i)}$.

Now the covariant differentiation of a vector is given by

$$T_{i;j}=\partial T_i/\partial x^j-T_h\Gamma_{ij}{}^h, \quad T_{i;(j)}=\partial T_i/\partial y^j,$$

where $\Gamma_{ij}{}^h$ is given by $\Gamma_{ij}{}^h=(\partial g_{i(k)}/\partial x^j)g^{h(k)}$.

For the covariant derivative of the relative metric tensor, we have

$$g_{i(j);k}=0, \quad g_{i(j);(k)}=0.$$

The author defines the curvature tensor through the formulas of Ricci:

$$u^i{}_{;j;k}-u^i{}_{;k;j}=0, \quad u^i{}_{;(j);(k)}-u^i{}_{;(k);(j)}=0,$$
$$u^i{}_{;j;(k)}-u^i{}_{;(k);j}=u^a S^i{}_{aj(k)},$$

and he derives the Bianchi's identities for this curvature tensor. If the curvature tensor S has the form

$$S_{i(j)k(l)}=\rho(g_{i(j)}g_{k(l)}+g_{k(j)}g_{i(l)}),$$

then ρ is a constant. Such a space is called space of constant relative curvature.

Now, for two points (x^i) and (x^i+dx^i), we have

$$[d(x, x+dx)]^2=g_{ij}(x)dx^i dx^j+\cdots,$$

where $g_{ij}(x)=g_{i(j)}(x, x)=-g_{i;j}(x, x)$. The Riemannian space V with the metric $ds^2=g_{ij}(x)dx^i dx^j$ is called the space associated with S. For the Christoffel symbols and the curvature tensor of this associated space, we have

$$\{j^i{}_k\}=\Gamma_{jk}{}^i(x, x)=\Gamma_{(j)(k)}{}^{(i)}(x, x),$$
$$R^i{}_{jkl}=S^i{}_{jk(l)}(x, x)-S^i{}_{jl(k)}(x, x).$$

If S is of constant relative curvature, then V is of constant curvature.

Take a vector $u^{(i)}(x_0)$ at a fixed point $(x_0) \in S$, and construct $v^i(x)=u^{(a)}g_{j(a)}(x, x_0)g^{ij}(x)$. We see that if $x=x_0$ then $u^i(x_0)=v^i(x_0)$, and if (x_0) and (x) are infinitesimally near, then the $u^i(x)$ is defined by Levi-Civita's parallel displacement from $v^i(x_0)$. Thus $v^i(x)$ is said to be obtained from $u^i(x_0)$ by a parallel displacement. This parallel displacement does not change the length of a vector if and only if $g_{i(a)(b)}=g_{i(a)}g_{j(b)}g^{ij}$. A space satisfying this condition is said to admit a parallelism.

Now consider a curve C in S. If the vectors at a fixed point x_0 which are obtained by parallel displacement of all tangent vectors of C are all the same, we call the curve C a geodesic with the center x_0. The equation of such a curve is given by

$$\frac{d^2x^i}{ds^2}+\Gamma_{jk}{}^i(x, x_0)\frac{dx^j}{ds}\frac{dx^k}{ds}=0.$$

In the last three sections, the author studies S admitting parallelism and squares of S. *K. Yano* (Paris)

306:

Tutaev, L. K. On differential geometry of a Riemannian space of two or three dimensions. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 65–77. (Russian)

307:

Nagarathnamma, H. S. Umbilical indicatrices of the unit tangent vector and the principal normal and binormal vectors of a curve in Riemannian space. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 351–359.

Let V_{n+1} be a Riemannian space with the metric $ds^2=a_{\alpha\beta}dy^\alpha dy^\beta$ and V_n a hypersurface in V_{n+1} given by $y^\alpha=y^\alpha(x^1, \cdots, x^n)$. The fundamental tensor of V_n is denoted by $g_{ij}=a_{\alpha\beta}y^\alpha{}_{;i}y^\beta{}_{;j}$, where $y^\alpha{}_{;i}=\partial y^\alpha/\partial x^i$. Consider a curve C: $x^i=x^i(s)$ in V_n and put $e^i=dx^i/ds$, $E^\alpha=y^\alpha{}_{;i}e^i$. Now, the author says, "Consider the locus of points whose coordinates are $(E^1, E^2, \cdots, E^{n+1})$. The locus of such points is a curve $\bar{C}$ in the hyperquadric Q_n defined by $a_{\alpha\beta}E^\alpha E^\beta=1$". But it seems to the reviewer that the definition of the curve $\bar{C}$ and that of Q_n can have geometrical meaning only when the enveloping space V_{n+1} is Euclidean. With this in mind, the author proves the following theorems: (1) The arc elements of $\bar{C}$ and C are connected by the relation $ds^2=(1/\rho^2)ds^2$, where $1/\rho$ is the first curvature of C in V_{n+1}. (2) The unit tangent vector to $\bar{C}$ is co-directional with the unit principal normal vector to C in V_{n+1}, and the ratio of torsion to curvature at any point of $\bar{C}$ is equal to the geodesic curvature of $\bar{C}$ at the corresponding point.

Let $\theta^\alpha=\theta_1{}^\alpha, \theta_2{}^\alpha, \cdots, \theta_n{}^\alpha$ be the unit principal normal vector and the first, the second, $\cdots$ binormal vectors at any point of C in V_{n+1}, defined by the Frenet formulas $\delta\theta_r{}^\alpha/\delta s=-\kappa_r\theta_{r-1}{}^\alpha+\kappa_{r+1}\theta_{r+1}{}^\alpha$ ($\theta_0{}^\alpha=E^\alpha$; $\kappa_{n+1}=0$) and $\kappa_1, \cdots, \kappa_n$ n curvatures corresponding to these vectors. The author considers the locus of a point whose coordinates are $\theta_r{}^\alpha$. She says that "the locus is a curve $\bar{C}_r$ in the hyperquadric Q_{r1n} defined by $a_{\alpha\beta}\theta_r{}^\alpha\theta_r{}^\beta=1$." But this can have a geometrical meaning only when V_{n+1} is Euclidean. The author then proves the following three theorems. (3) The arc elements of $\bar{C}_r$ and C are connected by the relation

$ds_r^2=(\kappa_r^2+\kappa_{r+1}^2)ds^2$. (4) The $(r-1)$st binormal vector to C in V_{n+1} is orthogonal to the unit tangent vector to $\bar{C}_r$. (5) If the unit principal normal vector to $\bar{C}_r$ is co-directional with the unit $(r-1)$st binormal vector to $\bar{C}$, then $\bar{C}_r$, the image of C, is a geodesic and conversely.

K. Yano (Paris)

308:

Prvanović, Mileva. Sur quelques formules de la géométrie conforme du sous-espace. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 53–66.

Let V_n be a subspace of a Riemann space V_m and K_p a linear vector space normal to V_n but not coincident with the first osculating vector space of V_n. The author defines a new differentiation operation which, when applied to the unitary tensors of K_p, yields tensors which are conformally invariant. Applying this operation in the usual manner, conformal analogues of the Frenet equations of V_n in V_m, relative to the vector field K_p, are obtained. Consideration of the integrability conditions of these Frenet equations leads to conformal analogues of the Gauss and Codazzi equations of V_n in V_m, relative to K_p. Special application is made to conformal curve theory.

A. Fialkow (Brooklyn, N.Y.)

309:

Pan, T. K. Relative first curvature and relative parallelism in a subspace of a Riemannian space. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 3–9.

Let V_n be a subspace of a Riemann space V_m. At each point $P(x^i)$ of V_n associate a set of $m-n$ unit vectors in V_m which are functions of x^i and dx^i at P and together with the tangent vector space of V_n at P form an m-dimensional vector space. With respect to the totality of these $m-n$ vectors, which generate $m-n$ congruences, relative first curvature, relative angular spread, pseudo-geodesics and relative parallelism are defined in V_n in V_m to generalize the concepts and results previously obtained by the author when $n=m-1$ [Canad. J. Math. **6** (1954), 210-216; MR **15**, 827].

A typical result is the following: The relative first curvature at a point P of a curve in V_n in V_m differs from its projected first curvature at P in V_n by the relative first curvature at P of the geodesic of V_n which is tangent to C at P.

A. Fialkow (Brooklyn, N.Y.)

310:

Katsurada, Yoshie. Alcune trasformazioni parallele di varietà algebriche $\{H, K\}$ di Del Pezzo-Segre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 719–725.

Let M_n be an affinely connected manifold of class C^A, imbedded in a projective space of sufficiently high dimension. Consider at any point $P \in M_n$ the differential elements E_B of order $B \leq A$; if H, K, are integers such that $0 \leq H < K \leq B$, take a differential element E_H, with center at P; the locus of the osculating S_K's at the E_K's of M_n through E_H is an algebraic variety $\{H, K\}$, of dimension $(K-H)n+H$, the so-called Del Pezzo-Segre variety. — The author applies the theory of extensors to these varieties, and studies the variety $\{H, K\}$ "parallel" to a given $\{H, K\}$: the word "parallel" means that the second $\{H, K\}$ is derived from the first one by considering the E_H parallel to the E_H which determines the first variety, the parallelism being uniquely defined by the given affine connection.

V. Dalla Volta (Rome)

311:

Ishihara, Shigeru. Groups of projective transformations and groups of conformal transformations. J. Math. Soc. Japan **9** (1957), 195–227.

The paper is divided into six sections. After a short introduction, the author discusses, in § 1, groups of projective transformations (p.t.) in a manifold M with a symmetric affine connection Γ. Let G be a group of p.t. of the affine connection Γ. If there exists a certain affine connection projectively related to Γ for which G is a group of affine transformations (a.t.), then the group G is said to be essentially affine with respect to the connection Γ. The author proves the following three theorems. (1) Let G be a compact group of p.t. of a manifold M with affine connection Γ. Then G is essentially affine with respect to Γ. (2) Let G be a transitive group of p.t. in a manifold M with affine connection Γ. If the isotropy group of G at a point of M is compact, then G is essentially affine with respect to Γ. (3) Let G be a transitive group of p.t. of a manifold M with affine connection Γ which is not projectively flat. If the identity component of the linear isotropy group of G at a point is irreducible, then G is essentially affine with respect to Γ.

In § 2, the author studies the groups of conformal transformations (c.t.) in a Riemannian manifold M with metric tensor g. Let G be a group of c.t. in M. If there exists a certain Riemannian metric conformally related to g which is invariant under G, then the group G is said to be essentially isometric with respect to g. He proves the following two theorems. (4) Let G be a compact group of c.t. in a Riemannian manifold M with the metric tensor g. Then G is essentially isometric with respect to g. (5) Let G be a transitive group of c.t. of a Riemannian manifold M with the metric tensor g. If the isotropy group of G at a point is compact, then G is essentially isometric with respect to g.

§ 3 is devoted to the discussions of groups of p.t. leaving the Ricci tensor invariant and contains the following two theorems. (6) Let M be a manifold with affine connection. If the homogeneous holonomy group (h.h.g.) of M has no invariant hyperplane, or if the restricted h.h.g. of M has no invariant covariant vector, then $P^*(M)=A(M)$. If, moreover, the Ricci tensor of M vanishes, then $P(M)=A(M)$. Here $P(M)$, $P^*(M)$ and $A(M)$ are, respectively, the group of all p.t., that of all p.t. which preserve the Ricci tensor and that of all a.t. of M. (7) If M is a complete affinely connected manifold, then $P^*(M)=A(M)$. If, moreover, the Ricci tensor of M vanishes, then $P(M)=A(M)$.

In § 4, the author studies groups of p.t. leaving the Ricci tensor in a Riemannian manifold invariant, and proves the following two theorems. (8) If the restricted h.h.g. of a complete Riemannian manifold M has no invariant vector, then $P_0^*(M)=I_0(M)$. If, moreover, the Ricci tensor of M vanishes, then $P_0(M)=I_0(M)$. Here $P_0(M)$, $P_0^*(M)$ and $I_0(M)$ are, respectively, the identity components of $P(M)$, $P^*(M)$ and $I(M)$, $I(M)$ being the group of all isometries in M. (9) If M is a compact Riemannian manifold, then $P_0^*(M)=I_0(M)$.

The author further studies, in § 5, groups of all c.t. leaving the Ricci tensor invariant. He proves: (10) If M is a complete, non-flat Riemannian manifold of n dimensions, then $C^*(M)=I(M)$ for $n>2$. If, moreover, the Ricci tensor of M vanishes, then $C(M)=I(M)$. Here $C(M)$ and $C^*(M)$ are, respectively, the group of all c.t. in M and that of all c.t. which preserves the Ricci tensor. (11) If M is a compact Riemannian manifold, then $C^*(M)=I(M)$. If, moreover, the Ricci tensor vanishes, then $C(M)=I(M)$.

The last Section is devoted to the proof of a lemma used in Section 1. *K. Yano* (Paris)

312:
Vrănceanu, G. G. La détermination des espaces à connexion affine localement euclidiens A_2 d'espèce trois. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957) no. 15, 43–49. (Romanian. French and Russian summaries)

When the affine connection is given in the form

$$\Gamma_{jk}{}^{i}=\pi_{jk}{}^{i}+\frac{2}{n+1}\delta_{(j}{}^{i}\Gamma_{k)},$$

where the $\pi_{jk}{}^{i}$ are projective components, then the characteristics are given by

$$-\pi_{22}{}^{1}dy^3-3\pi_{12}{}^{1}dy^2dx+3\pi_{21}{}^{2}dydx^2+\pi_{11}{}^{2}dx^3=0.$$

The "class 3" here investigated has coinciding characteristics [cf. O. Borůvka, Publ. Fac. Sci. Univ. Masaryk (1927) no. 85]. *D. J. Struik* (Cambridge, Mass.)

313:
Vrănceanu, G. G. Détermination des transformations ponctuelles d'espèce trois. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 447–456. (Romanian. Russian and French summaries)

In connection with the paper ≠ 312 above, the point transformations of an A_2 of the third class are determined in the case that the characteristic curves are parallel to one of the axes. In the case that they are not parallel but the $\Gamma_{ik}{}^{k}=0$, $i, k=1, 2$, the determination depends on an equation of Riccati.

D. J. Struik (Cambridge, Mass.)

314:
Vrănceanu, Gheorghe. Sur le groupe de stabilité d'un espace à connexion affine. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 121–124.

The stability group of an affine space A_n ($\Gamma_{jk}{}^{i}=\Gamma_{kj}{}^{i}$) is the group of automorphisms which preserves a point. Expressed in normal coordinates, this group is linear homogeneous. If A_n is not euclidean this group cannot be the full group with n^2 parameters; if the A_n is Riemannian, the group is orthogonal. If A_n is not euclidean, and has a transitive group, then the stability group cannot contain the special transformation $\sum x^i\partial_i f$.

D. J. Struik (Cambridge, Mass.)

315:
Dumitraş, Viorel. Sur les transformations du groupe de stabilité d'un espace à connexion affine A_n. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 14, 29–33. (Romanian. French and Russian summaries)

Vrănceanu, in the paper reviewed above, has shown that the stability group of an A_n, not euclidean, cannot contain the transformation $\sum x^i\partial_i f$. It is shown that also transformations of the form $\sum x^i\partial_i f+x^j\partial_j f$, $i\neq j$, and $a\sum x^i\partial_i f+x^1\partial_1 f$ ($a\neq 0, \pm 1, \pm\frac{1}{2}$) are excluded. When an A_3 with a transitive automorphism contains the transformation $x^1\partial_1 f+a\sum x^i\partial_i f$ ($a=1, \frac{1}{2}$), it is either euclidean or a space with maximal group.

D. J. Struik (Cambridge, Mass.)

316:
Dumitraş, Viorel. Sur le groupe de stabilité d'une espace à connexion affine. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957) no. 16, 49–52. (Romanian. French and Russian summaries)

Continuing the investigations of the previous paper [≠ 315 above], the author shows that the stability group of an A_n with a transitive automorphism, which is not affine euclidean, cannot contain the transformations $\sum x^\alpha\partial_{\alpha+1}f+\sum x^i\partial_i f$, $\alpha=1, \cdots, m$, $i=1, \cdots, n$, $m<n$.

D. J. Struik (Cambridge, Mass.)

317:
Takizawa, Seizi. On the induced connexions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 105–118.

Let $P(M, G)$ be a differentiable principal bundle and H be a subgroup of G such that G/H is reductive, i.e. there exists a canonical decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{f}$. If $H_1=H_1\times H_2$ (H_1 or H_2 may consist only of the identity element alone) is a direct decomposition of H, then $\mathfrak{g}$ decomposes into the form $\mathfrak{g}=\mathfrak{h}_1+\mathfrak{f}+\mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Lie algebras of H_1 and H_2 respectively. Let $E_1(E, H_1)$ be a principal bundle with base space E (=the associated bundle of $P(M, G)$ with fibre G/H) and group H_1 which is defined naturally from $P(M, G)$. The author defines the induced connection ω_1 on $E_1(E, H_1)$ of the connection $\tilde{\omega}$ of $P(M, G)$ by the projection $\tilde{\omega}_{\mathfrak{h}_1}$ of a $\mathfrak{g}$-valued form $\tilde{\omega}$ of $\mathfrak{g}$ onto $\mathfrak{h}_1$ in the above direct decomposition of $\mathfrak{g}$ and obtains the equations of Gauss-Codazzi-Ricci. The induced connection to the theory of submanifolds of Riemannian manifolds is applied to the canonical connections on universal bundles, the Stiefel-Whitney characteristic classes, and reductive Cartan connections. *A. Kawaguchi* (Sapporo)

318:
Kaganov, S. A. Geometry of space with a singular hyperareal metric. Mat. Sb. N.S. **42(84)** (1957), 497–512. (Russian)

Returning to the subject he has studied [Dok. Akad. Nauk SSSR (N.S.) **75** (1950), 487–490; MR **13**, 778] the author first builds up the foundations again and then proceeds to study necessary Legendre conditions and the second variation of the hyperplane of the extremal hypersurface in the space with the singular metric. This generalizes certain results obtained for $m=2$ by J. Radon, [Jber. Deutsch. Math. Verein. **47** (1937), 220–232].

D. J. Struik (Cambridge, Mass.)

PROBABILITY

See also 391, 409, 538, 681a-b, 781, 786, 788, 791, 793.

319:
Breny, H. Sur les fondements" de la théorie des probabilités. Rev. Questions Sci. (5) **19** (1958), 161–190.

The author argues in favour of a theory of probability regarded as a physical theory. He wishes to restrict the word "probability" to physical probability, but without accepting the long-run-frequency interpretation ("frequentism"). Like Borel [Probabilité et certitude, Presses Universitaires de France, Paris, 1950; MR **12**, 618] he considers that the only verifiable consequences of a probability model are those of very small probability ("Cournot's principle"), but he does not define "very small" as stringently as Borel did.

I. J. Good (Cheltenham)

320:
Onicescu, Octav. Sur les champs de vecteurs-somme. C. R. Acad. Sci. Paris **246** (1958), 3574–3576.

321:
Onicescu, Octav. Les probabilités sur une algèbre de Boole. C. R. Acad. Sci. Paris **246** (1958), 3210–3213.

322:

Nasr, Saad K. Sur l'unicité de la moyenne de Doss des variables aléatoires situées dans quelques espaces de Banach. Colloq. Math. **5** (1957), 85–94.

Doss a donné la définition suivante de l'espérance mathématique d'une variable aléatoire x à valeurs dans un espace métrique (D): On appellera espérance mathématique au sens de Doss de la variable aléatoire $x \in (D)$ tout élément $a \in (D)$ tel que

$$(a, \lambda) \leq E(x, \lambda) \quad \forall \lambda \in (D)$$

où (x, y) désigne la distance entre deux éléments quelconques x, y de (D) et E l'espérance mathématique classique d'une variable aléatoire à valeurs réelles.

L'auteur donne des conditions suffisantes assurant l'unicité de l'espérance mathématique de Doss dans les espaces de Banach suivants: (C): espace des fonctions continues dans [0, 1]; (c): espace des suites convergentes; (l): espace des suites $\{h_i\}$ telles que $\sum |h_i| < \infty$; (L): espace des functions sommables dans [0, 1].

A. Fuchs (Strasbourg)

323:

Steck, George P. A table for computing trivariate normal probabilities. Ann. Math. Statist. **29** (1958), 780–800.

Probabilities of arbitrary polyhedral regions under a general trivariate normal density can be found (accurate in the fourth decimal) by using the present table, a table by Owen [same Ann. **27** (1956), 1075–1090], and a table of the univariate normal distribution. {In the table headings change m to h.} *I. R. Savage* (Minneapolis, Minn.)

324:

Crow, Edwin L. A property of additively closed families of distributions. Ann. Math. Statist. **29** (1958), 892–897.

Consider a family of distribution functions, $F(x:\lambda)$, indexed by a real parameter λ which is "additively closed" under convolution. Three cases are considered, namely, λ is (i) any positive integer, (ii) any positive rational, and (iii) any positive real number. Sufficient conditions are given to insure that "the only linear combinations of a finite number of independent (random) variables with distributions in the family, $\sum_{r=1}^{k} c_r X_r$ ($c_r \neq 0$, real), whose distributions are also in the family are those with all $c_r = 1$." It may be shown that the assumption of continuity in λ of the characteristic function of $F(x:\lambda)$ which is added for case (iii) may be removed. Moreover, the assumption $F(x:\lambda) = 0$ for $x < 0$ and $F(x:\gamma) > 0$ for $x > 0$ may be replaced by $F(x:\lambda) = 0$ for $x < a\lambda$ and $F(x:\lambda) > 0$ for $x > a\lambda$ for some real a.

R. Pyke (New York, N.Y.)

325:

Durand, David; and Greenwood, J. Arthur. Random unit vectors. II. Usefulness of Gram-Charlier and related series in approximating distributions. Ann. Math. Statist. **28** (1957), 978–986.

Let $\xi_1, \cdots, \xi_n$ be independent random variables uniformly distributed on $(0, 2\pi)$, $V = \sum \cos \xi_i$, $W = \sum \sin \xi_i$, $R = (V^2 + W^2)^{\frac{1}{2}}$. Several approximations to the distributions and density functions of V, W, and R are considered, continuing work begun in the same Ann. **26** (1955), 233–246 [MR **16**, 1034]. The authors give calculations for n between 3 and 10, suggesting that in this range the normal approximation to the density of V is substantially improved by adding one term in an Edgeworth or Gram-Charlier series, but several further terms give little or no improvement. For $n \geq 5$, the one additional term appears to give 2-decimal accuracy for the density of V. *T. E. Harris* (Santa Monica, Calif.)

326:

Bodiou, Georges. Probabilités déterminées par leurs propositions presque certaines. C. R. Acad. Sci. Paris **246** (1958), 3000–3002.

327:

Longuet-Higgins, M. S. On the intervals between successive zeros of a random function. Proc. Roy. Soc. London. Ser. A. **246** (1958), 99–118.

Let f be a stationary Gaussian process, let τ be the interval between two successive zeros of f, with a probability density $p(\tau)$, and let $U(\tau)$ be the probability that f be positive over an interval of length τ. It is shown that $p(\tau) = (2/N_0) d^2U/d\tau^2$, with N_0 the average number of zeros per unit time. This reduces the problem of finding $p(\tau)$ to that of finding $U(\tau)$. $U(\tau)$ is approximated as follows: n points $t_1, \cdots, t_n$ are chosen in an interval of length τ and $U_n(t_1, \cdots, t_n)$, defined as prob$\{f(t_1) > 0, \cdots, f(t_n) > 0\}$, approximates $U(\tau)$. It is claimed that if f is continuous, then it is reasonable to assume that $U_n \to U$ as $n \to \infty$ and $\max(t_i - t_{i-1}) \to 0$. The measure-theoretical foundation of this claim is not considered here; it can presumably be supplied by a separability condition, or by a direct construction of the probability measure on a space of continuous functions. A simple formula for U_n is given in terms of the region S_n of the unit n-sphere bounded by hyperplanes $\sum a_{ij}\eta_j = 0$. The approximation U_n gives rise to two approximations p_n and p_n^* to p; analysis, graphs, and comparison with experiment indicate that the approximations are already fairly accurate for $n \sim 5$ or 6.

V. E. Beneš (Murray Hill, N.J.)

328:

Rao, J. N. K. A characterization of the normal distribution. Ann. Math. Statist. **29** (1958), 914–919.

The author gives a characterization of the normal distribution. The characterization of the univariate normal distribution (Theorem 1) is a particular case of theorem 6.1 in this reviewer's paper [Proceedings of the third Berkeley symposium on mathematical statistics and probability, 1954–1955, vol. II, pp. 195–214; MR **18**, 942]. He then extends this result to the multivariate case. The sufficiency of his condition is a special case of a result of R. G. Laha [Sankhyā **14** (1955), 367–368; MR **16**, 940], while the necessity is obtained by generalizing a result of J. Daly [Ann. Math. Statist. **17** (1946), 71–74; MR **7**, 464]. *E. Lukacs* (Washington, D.C.)

329:

Laha, R. G. On a factorization theorem in the theory of analytic characteristic functions. Ann. Math. Statist. **29** (1958), 922–926.

The author proves the following theorem: Let $\phi_j(t)$, $j = 1, \cdots, n$, be characteristic functions (c.f.'s) of non-degenerate distributions, $\phi(t)$ be an entire c.f. of order $\rho < \infty$, and $\alpha_1, \cdots, \alpha_n$ be positive numbers. Suppose that $\phi(t) = \prod_{j=1}^{n} \{\phi_j(t)\}^{\alpha_j}$ for all real t in $|t| < \delta$, $\delta > 0$. Then the $\phi_j(t)$, $j = 1, \cdots, n$, are entire functions of finite order $\leq \rho$.

J. Wolfowitz (Ithaca, N.Y.)

330:

Linnik, Yu. V. Some theorems on the factorization of infinitely divisible laws. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 549–551. (Russian)

If a probability distribution is the convolution of two distributions, the latter are called components of the given

distribution. Let I_0 be the class of those infinitely divisible distributions which have only infinitely divisible components. If ϕ is the characteristic function of an infinitely divisible distribution, the Lévy formula expresses $\log \phi$ as the sum of a polynomial of degree $\leqq 2$ and an integral over $(-\infty, \infty)$, the latter being a generalized sum of Poisson components. If this integral is extended only over $(0, \infty)$, or over a bounded set, or $\cdots$, the distribution corresponding to ϕ is said to have a Poisson spectrum that is positive, or bounded, or $\cdots$. Thus, if ϕ is the characteristic function of an infinitely divisible distribution with a finite or countably infinite Poisson spectrum, $\log \phi$ has the form

$$(1) \qquad \beta_1 it - \gamma t^2 + \sum_m \lambda_m (e^{it\mu_m} - 1), \quad \lambda_m > 0,$$

and ϕ is said to have a rational spectrum if μ_m/μ_l is rational, for all l, m. The following results are announced. (a) If ϕ is the characteristic function of an infinitely divisible distribution with a positive rational Poisson spectrum, with $\gamma > 0$, then, if $\phi \in I_0$, μ_m in (1) must run through numbers of the form

$$(2) \qquad \cdots, \ k_{-2}k_{-1}\mu, \ k_{-1}\mu, \ \mu, \ \mu k_1^{-1}, \ \mu k_1^{-1}k_2^{-1}, \ \cdots,$$

where $\mu > 0$, and each k_i is a natural number (repetition allowed). The converse is true if the Poisson spectrum is also supposed bounded, and in this case the terms to the left of μ in (2) do not appear. (b) If an infinitely divisible distribution has a bounded positive Poisson spectrum confined to the interval $(0, a]$, the log of the characteristic function of any component is given by

$$(3) \qquad P_3(it) + t^4 \int_0^a e^{itu} f(u)\, du,$$

where P_3 is a polynomial of degree at most 3 and f is real with summable square. (c) If in (b) the spectrum is bounded, and is rational and finite to the right of some point a, and if there is a Gaussian component, the logarithm of the characteristic function of any component is given by the sum of (3) and a sum like that in (1) where, however, μ_m is now a rational multiple of some μ, and λ_m may be complex. If there is no Gaussian component, a related form is found for the characteristic function of any component.

Cramér [Math. Z. **41** (1936), 405–414] has shown that a Gaussian distribution has only Gaussian components. Raikov [Izv. Akad. Nauk SSSR Ser. Mat. **1** (1938), 91–124] has shown that a Poisson distribution has only Poisson components. Linnik [Teor. Veroyatnost. i Primenen **2** (1957), 34–59; MR **19**, 777] has shown that the convolution of a Gaussian and Poisson distribution has components only of the same type. The present results, together with those reviewed just below thus go far beyond earlier work. *J. L. Doob* (Urbana, Ill.)

331:

Linnik, Yu. V. On the factorization of infinitely divisible laws. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 735–737. (Russian)

[For the notation and references see the preceding review]. The following results are announced. (A) If an infinitely divisible distribution with a Gaussian component is in I_0, its Poisson spectrum must be finite or countably infinite and its positive [negative] Poisson spectrum must have the form (2) in the preceding review, with $\mu > 0$ [$\mu < 0$]. If the spectrum is bounded, these conditions are also sufficient. This result is stable in the sense that if a distribution is close to one with a bounded Poisson spectrum of the form described, its components will be close to infinitely divisible distributions with the same Poisson spectrum. (B) Let $\{t_k, k \geqq 1\}$ be a sequence of real (presumably non-zero) numbers converging to 0, let α_k be strictly positive, let ϕ_j be a characteristic function, and let ϕ be the characteristic function of an infinitely divisible distribution with a countable bounded Poisson spectrum. Suppose that, for all k,

$$\phi_1(t_k)^{\alpha_1} \cdots \phi_s(t_k)^{\alpha_s} = \phi(t_k).$$

Then each ϕ_j has a countable Poisson spectrum, a subset of that of ϕ. *J. L. Doob* (Urbana, Ill.)

332:

Prékopa, András. On the convergence of infinite series of independent random variables. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 191–198. (Hungarian)

Let $\xi_1, \xi_2, \cdots$ be a sequence of independent random variables and let $f[\gamma]$ be the set of finite subsets [of all subsets] of the positive integers. Write $\xi(A) = \sum \xi_k$, where the summation is extended over all integers $k \in A$ provided that, in case A is infinite, $\sum \xi_k$ converges with probability one, regardless of the order of summation. Denote by $F(x, A)$ the distribution function of $\xi(A)$. The author shows that the compactness of the set $\{F(x, A), A \in f\}$ is sufficient to insure that the series (1) $\sum_{k=1}^{\infty} \xi_k$ should converge with probability one regardless of the order of summation. Conversely, this property of the series (1) insures the compactness of the set $\{F(x, A), A \in \gamma\}$. He also gives a second condition for the convergence of the series (1), expressed in terms of the quantiles of $\xi(A)$. [An English version of this paper appeared in Publ. Math. Debrecen **4** (1956), 410–417; MR **19**, 891]. *E. Lukacs* (Washington, D.C.)

333:

Barra, Jean-René. Lois des grands nombres pour des suites doubles de variables aléatoires. C. R. Acad. Sci. Paris **246** (1958), 2999–3000.

334:

Rihter, V. Mehrdimensionale lokale Grenzwertsätze für grosse Abweichungen. Teor. Veroyatnost. i Primenen. **3** (1958), 107–114. (Russian. German summary)

Let $X^{(1)}, X^{(2)}, \cdots$ be a sequence of independent, identically distributed m-dimensional random vectors with zero means, $Z_n = (X^{(1)} + \cdots + X^{(n)})n^{-\frac{1}{2}}$. Assuming that the probability density $p_{Z_n}(x_1, \cdots, x_m)$ of Z_n exists and is bounded for n large enough and that the moment-generating function of $X^{(1)}$ exists in a neighborhood of the origin, an asymptotic expression for p_{Z_n}, valid for $|x_j| > 1$ and $|x_j| = o(n^{\frac{1}{2}})$ as $n \to \infty$, $j = 1, \cdots, m$, is derived. An analogous expression is obtained for the elementary probability law of $Z_{(n)}$ when $X^{(1)}$ has a lattice distribution. The case $m = 1$ was treated by the author in Teor. Veroyatnost. i Primenen. **2** (1957), 214–229 [MR **19**, 1087]. *W. Hoeffding* (Stanford, Calif.)

335:

Chernoff, H.; and Teicher, H. A central limit theorem for sums of interchangeable random variables. Ann. Math. Statist. **29** (1958), 118–130.

A triangular array of random variables X_{nk}, $k = 1, \cdots, k_n (\to \infty)$, $n = 1, 2, \cdots$ is considered. Every subcollection of the random variables in the nth row X_{nk}, $k = 1, 2, \cdots, k_n$, is assumed to have a joint distribution symmetric in its arguments (the authors call this property interchangeability). Let $m_n \to \infty$ so that $m_n/k_n \to \alpha$, $0 < \alpha < 1$. Under the assumption that the row sums $\sum_k X_{nk}$

are zero, $EX_{nk}^2 \equiv 1$, $\max_{1 \le k \le k_n} |X_{nk}|/\sqrt{k_n} \to 0$ and $\sum_k X_{nk}^2/k_n \to 1$ in probability, $\sum_{k=1}^{m_n} X_{nk}/\sqrt{m_n}$ is shown to be asymptotically normally distributed.

M. Rosenblatt (Bloomington, Ind.)

336:

Rihter, Vol'fgang. Local limit theorems for large deviations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 53–56. (Russian)

The author reports the results, with proofs omitted, from his paper of the same title [Teor. Veroyatnost i Primenen. **2** (1957), 214–229; MR **19**, 1087].

H. P. Edmundson (Santa Monica, Calif.)

337:

Petrov, V. V. A local theorem for latticed distributions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 49–52. (Russian)

Let $X_1, X_2, \cdots$ be mutually independent integral-valued random variables, and let A_n, s_n^2 be, respectively, the expectation and variance of $x_n = \sum_1^n X_j$. Define

$$L_n = s_n^{-2-\delta} \sum_{m=1}^{n} E\{|X_m - E\{X_m\}|^{2+\delta}\},$$

where δ is a constant, $0 < \delta \le 1$. The author proves that

$$|s_n P\{x_n = N\} - (2\pi)^{-1} \exp\{-(N-A_n)^2/2s_n^2\}| \le CL_n,$$

where C does not depend on n or N, under the following hypotheses. (α) The highest common factor of the integers j with $(1/\log n)\sum_{m=1}^{n} P\{X_m = 0\}P\{X_m = j\} \to \infty$ is 1. (It is assumed that X_m is normalized in such a way that the second factor in the sum is a maximum when $j = 0$.) (β) $s_n \to \infty$ and $\sup_n L_n s_n^{\delta} < \infty$. Other results on the local normal law limit theorems for not necessarily identically distributed integral-valued summands have been obtained recently by Freiman [Vestnik Leningrad. Univ. **11** (1956), no. 1, 57–73; MR **17**, 1096] and Prohorov [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 535–538; MR **16**, 494].

J. L. Doob (Urbana, Ill.)

338:

Prohorov, Yu. V.; and Fiš, M. [Fisz, M.]. A characterisation of normal distributions in Hilbert space. Teor. Veroyatnost. i Primenen. **2** (1957), 475–477. (Russian. English summary)

If ξ is a random element of Hilbert space H, its mathematical expectation $E\xi$ is an element of H such that, for each $f \in H$, $E(f, \xi) = (f, E\xi)$. If two random elements $\xi \in H$ and $\eta \in H$ have the same probability distribution P this fact is denoted by $\xi \sim \eta$. Employing conditions (α) $E\|\xi\|^2 < \infty$, (β) $E(f, \xi)^2 > 0$ for all $f \in H$, $f \ne \theta$, (γ) $E\xi = \theta$, where θ is the null element of H, the following generalization of a theorem of Polya is proved.

Suppose that $\xi^{(1)}$, $\xi^{(2)}$, $\xi^{(3)}$ are random elements of H satisfying conditions (α), (β), (γ) and that $\xi^{(1)} \sim \xi^{(2)} \sim \xi^{(3)}$. Furthermore, suppose that (δ) $\xi^{(1)}$ and $\xi^{(2)}$ are independent, (ε) $\xi^{(1)} + \xi^{(2)} \sim A\xi^{(3)}$, where A is a linear, bounded, self-adjoint positive operator on H. Then the distribution P of each of the elements $\xi^{(i)}$ is normal and $A = 2^{\frac{1}{2}}I$, where I is the identity operator.

H. P. Kramer (Murray Hill, N.J.)

339:

Dawson, Reed; and Good, I. J. Exact Markov probabilities from oriented linear graphs. Ann. Math. Statist. **28** (1957), 946–956.

The reviewer's result for the distribution of transition totals in a sample from a Markov chain [J. Roy. Statist. Soc. Ser. B **17** (1955), 235–242; MR **17**, 982] is generalised to give the conditional distribution of m-tuples in a circular sequence, given the n-tuple count ($n < m$), and that the sequence is Markovian of order less than n. The method of proof is quite different, the authors' being based on some recent results in the theory of graphs [Tutte, W. T. and Smith, C. A. B., Amer. Math. Monthly **48** (1941), 233–237; van Aardenne-Ehrenfest, T. and de Bruijn, N. G., Simon Stevin **28** (1951), 203–217; MR **13**, 857].

The authors show that for long sequences the conditional distribution is equivalent to the distribution of cell-frequencies in a contingency table with fixed marginal totals and independent attributes. An exact formula is given for the factorial moments of pair frequencies in a circular sequence, conditional on given single-frequencies.

P. Whittle (Wellington)

340:

Nagaev, S. V. Some limit theorems for homogeneous Markoff chains. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 237–239. (Russian)

It is assumed that the transition functions $p^{(n)}(\eta, A)$ approach a stationary limit exponentially fast and uniformly in η and A. Conditions are given for the central limit theorem and for a limiting stable distribution of the successive sums of a functional of the chain. In the case of a countable state space local limit theorems are given. There are no proofs but it is stated that the spectral theory of linear operations is applied.

K. L. Chung (Syracuse, N.Y.)

341:

Kemeny, John G.; and Snell, J. Laurie. Semimartingales of Markov chains. Ann. Math. Statist. **29** (1958), 143–154.

If P is a transition matrix, the vector z is called an upper semimartingale, lower semimartingale or martingale according as $Pz \ge z$, $\le z$, or $= z$, respectively. The authors assume that P has a non-empty set B of absorbing states. The set U of all non-negative upper semimartingales with prescribed boundary values $\{v_j : j \in B\}$ is the set of all vectors z satisfying (i) $Pz \ge z$, (ii) $z \ge 0$, and (iii) $\{z\}_j = v_j$, $j \in B$. U is shown to be a convex polyhedron of dimension equal to the number of non-absorbing states. (Geometrical techniques and interpretations are used throughout.) A representation theorem for elements of U is obtained in which the elements are expressed as convex combinations of martingales of certain related Markov chains. Similar results for lower semimartingales are also derived. These results are applied to some sequential games and to discrete subharmonic functions, yielding in the latter case a representation of such a function as a convex combination of certain harmonic functions.

R. Pyke (New York, N.Y.)

342:

Blackwell, David. Another countable Markov process with only instantaneous states. Ann. Math. Statist. **29** (1958), 313–316.

Given a Markov chain with stationary transition probabilities $p(i, j, t) : i, j = 1, 2, 3, \cdots, t \ge 0$, such that

$$\lim_{t \downarrow 0} p(i, i, t) = 1, \quad i = 1, 2, 3, \cdots;$$

the state i is said to be instantaneous if

$$\lim_{t \downarrow 0} t^{-1}[1 - p(i, i, t)] = +\infty.$$

Chains with all states instantaneous were constructed by R. Dobrusin [Teor. Veroyatnost. i Primenen. **1** (1956), 481–485; MR **19**, 691] and by W. Feller and H. P. McKean, Jr., [Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 351–354; MR **19**, 327]. Here is a new, simpler, and better example.

H. P. McKean, Jr. (Cambridge, Mass.)

343:

Maslov, K. V.; and Povzner, A. Ya. On infinitesimal operators of a class of Markov processes. Teor. Veroyatnost. i Primenen. 3 (1958), 70–83. (Russian. English summary)

Consider a Markov process with transition probabilities given by $F(t, x; \tau, y)$. Let

$$T_t^f = \int_{-\infty}^{\infty} f(y) d_y F(t, x; \tau, y),$$

$$K_f(t, x) = \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{f(y)-f(x)}{\delta} d_y F(t-\delta, x; t, y),$$

when this limit exists. Then under certain assumptions on the transition probabilities it is shown that this infinitesimal operator is given by

$$K_f(t, x) = \int_{-\infty}^{\infty} \{f(x+y)-f(x)-\frac{y}{1+y^2} f'(x)\} \frac{1+y^2}{y^2} d_y \sigma(t, x, y) + \gamma(t, x) f'(x),$$

where $\sigma(t, x, y)$ is a monotone function of y of bounded variation. The assumptions made are of the following type: (A) for $f(x)$ bounded and continuous, for fixed t and x, $T_{t-\delta}^{\delta} f$ approaches $f(x)$ as $\delta \to 0$, (B) a condition which assures that the domain of K_f has sufficiently many smooth functions, (C) $\int_{|y-x|>N} d_y \delta^{-1} F(t-\delta, x; t, y) \to 0$ as $N \to \infty$ uniformly in δ. Special cases of interest are: (a) $\int_{|y-x|\geq \varepsilon} d_y \delta^{-1} F(t-\delta, x; t, y) \to 0$ as $\delta \to 0$ for any ε, if and only if σ (as a function of y) has a jump at 0 and is constant elsewhere (diffusion type process), and (b) $\int_{x-a}^{x+a} d_y \delta^{-1} F(t-\delta, x; t, y) < C$ for fixed t and x and all δ if and only if $\int_{-\infty}^{\infty} (1+y^2) y^{-2} d_y \sigma(t, x, y) < \infty$ (purely discontinuous type process). Properties of the path functions are not discussed. The generalization to n dimensions is discussed. {In formula (22) replace $y^2/(1+y^2)$ by $(1+y^2)/y^2$.}

J. L. Snell (Hanover, N.H.)

344:

Moyal, J. E. Discontinuous Markoff processes. Acta Math. 98 (1957), 221–264.

Consider a non-homogeneous Markov process $\xi(t)$ and a sequence of jump times $0=\eta_0<\eta_1<\cdots$. Let x, X denote resp. a point and a set in the state space, $n \geq 0$,

$$\chi(X, t|x_0, t_0) = \Pr\{\xi(t) \in X | \xi(t_0) = x_0\},$$
$$\chi_n(X, t|x_0, t_0) = \Pr\{\eta_n \leq t < \eta_{n+1} | \xi(t_0) = x_0\},$$
$$\psi_n(X, t|x_0, t_0) = \Pr\{\eta_n \leq t; \xi(\eta_n + 0) \in X | \xi(t_0) = x_0\}.$$

We have then

$$(*) \quad \chi(X, t|x_0, t_0) = \chi_0(X, t|x_0, t_0) + \int \chi(X, t|x, \tau) \psi_1(dx, d\tau|x_0, t_0).$$

Noting that χ_0 is the probability of transition without jump, we see that the well-known development by Feller [Trans. Amer. Math. Soc. 48 (1940), 488–515; MR 2, 101] corresponds to the case where there is no change of state without a jump, so that χ_0 is of a specifiable form. Observe that these so-called "jumps" may be of a prescribed kind and need not include all ordinary jumps (of the sample function). The main point of the paper is to show that part of Feller's theory can be carried over painlessly (except for notations and formalities) to the present generality. E.g., we have $\chi_n = \chi_j * \psi_{n-j}$ in understandable notation, and $\chi_R = \sum_{n=0}^{\infty} \chi_n$ can be obtained by the same iterative procedure and represents a solution of (*) except that it may be "incomplete". In the latter case not all possible transition is via a finite number of jumps and Doob's continuation procedure [Trans. Amer. Math. Soc. 58 (1945), 455–473; MR 7, 210; this paper is not cited] can also be applied, etc. Because the presentation is analytical the scope of processes covered is not delineated.

K. L. Chung (Syracuse, N.Y.)

345:

Dynkin, E. B. Markov jump processes. Teor. Veroyatnost. i Primenen. 3 (1958), 41–60. (Russian. English summary)

A Markov process $x(t, \omega)$ ($t \geq 0$, $\omega \in \Omega$) on a measurable space ($\mathscr{E}$, $\mathfrak{B}$) is called a jump process, if for every $\omega \in \Omega$ and $t \geq 0$ there exists an $\varepsilon > 0$ such that $x(t, \omega) = x(t+h, \omega)$ for all $h \in [0, \varepsilon)$. The author determines the infinitesimal operators of all Markov jump processes. Special classes of jump processes have been studied by Feller [Math. Ann. 113 (1936), 113–160; Trans. Amer. Math. Soc. 48 (1940), 488–515; MR 2, 101], Doeblin [Scand. Aktuarietidskr. 1939, 211–222; MR 1, 247], Dubrovskii [Dokl. Akad. Nauk SSSR 19 (1938), 439–444; Izv. Akad. Nauk SSSR Ser. Mat. 8 (1944), 107–128; MR 6, 160] and Doob [Trans. Amer. Math. Soc. 58 (1945), 455–473; MR 7, 210].

H. P. Edmundson (Santa Monica, Calif.)

346:

Čerkasov, I. D. On transforming the diffusion process to a Wiener process. Teor. Veroyatnost. i Primenen. 2 (1957), 384–388. (Russian. English summary)

The author gives necessary and sufficient conditions under which there exists a one-to-one transformation of coordinates by which the diffusion equation

$$(1) \qquad \partial f/\partial t + a(t, x) \partial^2 f/\partial x^2 + b(t, x) \partial f/\partial x = 0$$

is carried into the equation

$$(2) \qquad \partial f/\partial t + \partial^2 f/\partial x'^2 = 0,$$

and that solution of (1) which represents a transition probability density is mapped into the corresponding solution of (2) which is the transition probability density of the Wiener process. The method used is elementary and the conditions are: (a) a bounded and $a > 0$; (b) a_{xxx}, b_{xx}, a_{txx}, etc. exist; (c) $W(\alpha, \beta, \gamma) = 0$ where W is the Wronskian and $\alpha = a^{\frac{1}{4}}$, $\beta = a^{\frac{1}{4}} \int_0^x a(t, y)^{-\frac{1}{4}} dy$, $\gamma = 2b - a_x - a^{\frac{1}{4}} \int_0^x a_t a^{-3/2}(t, y) dy$.

H. P. Kramer (Murray Hill, N.J.)

347:

Rosenblatt, Murray. Some purely deterministic processes. J. Math. Mech. 6 (1957), 801–810.

Deterministic processes having prediction error variance $V=0$ if the prediction is based on the entire past, the author forms predictions based on a finite interval of length n and studies the speed of $V \to 0$ as $n \to \infty$. Discrete processes with spectral density zero in an interval give $V = O(a^n)$, $0 < a < 1$, and a special process where the density is positive with high order zero contact gives $V = O(n^{-b})$, $b > 0$. The methods exploit G. Szegö [Orthogonal polynomials, Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939; MR 1, 14].

H. Wold (New York, N.Y.)

348:

Sack, R. A. Restricted random walks and the use of moments. Phil. Mag. (8) 3 (1958), 504–507.

For a random walk on certain crystal lattices with more than one atom per unit cell, not all sites are equivalent. Diffusion on such a lattice was treated by Hammersley [Compositio Math. 11 (1953), 171–186; MR 19, 71] for quite general types of lattices using the theory of Markov chains. It is shown here, however, for special types of lattices, for example the hexagonal close-packed lattice, that despite the inequivalence of sites, the first

and second moments of the displacement are the same for all sites and one can, therefore, calculate the diffusion constant very easily from the universal values of the second moments. *G. Newell* (Stockholm)

349:

Sherman, B. The limiting distribution of Brownian motion in a bounded region with instantaneous return. Ann. Math. Statist. **29** (1958), 267–273.

The author considers a process in which a point moves in a Brownian fashion when inside a region D, and returns instantaneously to the interior of D in accordance with a prescribed law whenever it reaches the boundary. The existence of a limiting distribution is proved, and the distribution related to the Green's function of a potential problem on D. The proof utilises the imbedded Markov chain associated with arrivals at the boundary.

P. Whittle (Wellington)

350:

Clarke, A. Bruce. Maximum likelihood estimates in a simple queue. Ann. Math. Statist. **28** (1957), 1036–1040.

The queue in question is that formed before a single server with exponential distribution of service time by a traffic input of Poisson character. The maximum likelihood estimates are those of the input rate, λ, and of the service rate, μ, for the equilibrium condition: $\lambda/\mu<1$. They are obtained from observations made during a period during which the server is busy for a fixed time τ; the observations are of (i) v, the initial queue size, (ii) m, the total number of departures (service completions), (iii) T, the time of the last departure, and (iv) n, the total number of arrivals up to T. The consideration of busy time rather than total time is introduced to simplify the likelihood function. The maximum likelihood estimates $\hat{\lambda}$ and $\hat{\mu}$ have the following simple approximations: $\hat{\lambda}=(n+v)/T$, $\hat{\mu}=(m-v)/\tau$. {Reviewer's note: There seems to be a misprint in the second equation of section 4, which should read $\hat{\lambda}=(\hat{\lambda}-\hat{\mu})(m-v-\hat{\mu}\tau)$.}

J. Riordan (New York, N.Y.)

351:

Girault, Maurice. Files d'attente. Loi de survie d'un intervalle à partir d'un instant quelconque. C. R. Acad. Sci. Paris **246** (1958), 2838–2839.

The relation between the interval V formed by a randomly chosen point in a service interval (of a traffic system) and the point of service completion, and the service interval itself is expressed in terms of the corresponding characteristic functions by $\varphi_V(x)=(\varphi(z)-1)/z\varphi'(0)$, the prime indicating a derivative. This is in agreement with the known result [C. Palm, Ericsson Technics no. 44 (1943), MR 6, 160] that the complementary distribution function $G_e(t)=1-G(t)$ of V is given by $G_e(t)=\int_t^\infty F_e(t)dt/\int_0^\infty F_e(t)dt$, with $F(t)$ the service time distribution. It is noted that $\varphi(z)$ is the characteristic function of an (almost surely) nonnegative random variable with finite mean, and that V is also nonnegative and unimodal. This leads to the extension that if $\varphi(z)$ is the characteristic function of a random variable with a moment of order k, and either k is even or the random variable is almost surely nonnegative, then

$$\Phi(z)=(\varphi^{(k-1)}(z)-\varphi^{(k-1)}(0))/z\varphi^{(k)}(0),$$

with the exponents indicating derivatives of the corresponding order, is a characteristic function.

J. Riordan (New York, N.Y.)

352:

Galliher, Herbert P.; and Wheeler, R. Clyde. Nonstationary queuing probabilities for landing congestion of aircraft. Operations Res. **6** (1958), 264–275.

The authors consider the problem of landing congestion of aircraft at an airport. The model assumes that planes arrive with a Poisson distribution, but that the average arrival rate can vary during the day. It is assumed that the servicing time is constant (although suggestions are given to handle the varying but non-random case also). The day is divided into intervals of length equal to the service time, and it is assumed that within any interval the average arrival rate is fixed although it can vary from interval to interval. The number of servers may also vary only from interval to interval. The authors present a method for computing the state probabilities at the end of each interval and from this the mean number in the system, or waiting. Actual distributions are obtained for a single-server system with constant service time and a variation in arrival rates based on observations of the New York terminal area. Results are given for several service times and several peak arrival rates.

H. M. Gurk (Princeton, N.J.)

353:

Pollack, M. Some studies on shuttle and assembly-line processes. Naval Res. Logist. Quart. **5** (1958), 125–136.

The author derives recurrence relations for the delay and arrival times associated with various types of multistage shuttle and assembly lines. In particular, it is shown that there is an analogy between these two types of processes so that results derived for one type can be used to obtain results concerning the other type.

R. Bellman (Santa Monica, Calif.)

354:

Reich, Edgar. On the integrodifferential equation of Takács. I. Ann. Math. Statist. **29** (1958), 563–570.

The queueing model having a single service in order of arrival, Poisson arrivals at a general rate $\lambda(t)$, and independent service times with distribution $H(\cdot)$ is considered. The integrodifferential equation, due to Takács, for the distribution of the mixed Markov process $\eta(t)$ (=waiting-time of someone arriving at t) is solved in a new manner taking advantage of the regularity properties of transforms. The distribution of $\eta(t)$ can be determined from the above quantities if $\Pr\{\eta(t)=0\}$ is known; the latter satisfies a Volterra equation of the first kind. It is shown that under suitable assumptions of absolute continuity and integrability (L_2), the Volterra equation can be differentiated to give a Volterra equation of the second kind, which has a unique solution. The kernel and the forcing function in the Volterra equation of first kind are given only as inversion integrals, and so their simple intuitive probabilistic significance is lost to the reader.

The paper concludes by discussing the asymptotic properties of $\eta(t)$ as $t\to\infty$ without any restrictions on the arrival process or the service times. The principal result is that $\eta(t)=o(t)$ as $t\to\infty$, under reasonable conditions which amount to not overloading the system.

V. E. Beneš (Murray Hill, N.J.)

355:

Swensson, Olle. An approach to a class of queuing problems. Operations Res. **6** (1958), 276–292.

The author considers a rather interesting class of waiting-line problems. A finite number of customers make calls for service on a system with several servers. If no server is free then the customer waits until one is available. Each customer may call for additional service either while

he is being served or while he is waiting for service. The arrivals of original calls, as well as those of calls for extra service, have Poisson distributions. The author allows for different rates for initial calls and the two types of extra service calls, but for the most part considers special cases in which at least two of the three rates are equal. Service time is also Poisson for each call for service with different rates for initial and extra service (again he considers mainly the special, equal-rate case). Two types of service procedure are examined — one in which calls are processed in order of their arrival, the other in which service is given in order of the arrival of the initial calls of the customers. Approximate solutions are obtained for both types. {The author's development of the assumptions leading to the approximations, however, is not clearly justified, in the reviewer's opinion. The basis for the choice of parameters in the state equations used to obtain the "solution" seems rather abruptly and unconvincingly obtained.}

H. M. Gurk (Princeton, N.J.)

356:

Stoller, David S. Some queuing problems in machine maintenance. Naval Res. Logist. Quart. **5** (1958), 83–87.

This paper gives a quite general formulation of the machine maintenance problem with time independent service and failure rates. It is assumed that there are several kinds of failures and that the failure and repair rates may be arbitrary functions of the state of the system (number of machines that require various types of repair). Explicit solutions are given for some cases in which the rate of failure depends only upon the number of machines that require no repair, and the rate of repairs of type j depends only upon the number requiring repairs of type j.

G. Newell (Stockholm)

357:

Urbanik, K. On a stochastic model of a cascade. Studia Math. **16** (1958), 237–267.

Consider a temporally homogeneous Markov process with a continuous time parameter and denumerable states 0, 1, ···. If the "intensity" of transition $a_n^{(k)}$ from state n to state k has the form $a_n^{(k)}=na_1^{(k-n+1)}$, then the state at time t may be interpreted as the size of a population, where $a_1^{(j)}dt$ is the probability that one object existing at t is transformed into j objects in $(t, t+dt)$, $j=0, 1, \cdots$, and different objects are independent. However, the Markov process formulation does not consider individual lines of descent; e.g., the answer to the question "what is the time of death of the second son of the third son" is not contained in the Markovian model. The author defines what amounts to a space of family trees, and postulates a measure satisfying properties corresponding to the independence of different branches, temporal homogeneity, and Markovian character. He then shows that every such system defines a Markovian population process, and every Markovian population process can be obtained from such a system. The paper gives proofs for results announced earlier [Bull. Acad. Polon. Sci. Cl. III, **3** (1955), 349–351; MR **17**, 276.] {Measures on spaces of family trees were used for the Galton-Watson branching process by, e.g., R. Otter [Ann. Math. Statist. **20** (1949), 206–224; MR **11**, 41] and C. J. Everett and S. Ulam [Los Alamos Declassified Documents 533 and 534 (1948)]; for certain age-dependent processes by R. Bellman and T. Harris [Ann. of Math. (2) **55** (1952), 280–295; MR **13**, 664].}

T. E. Harris (Santa Monica, Calif.)

358:

Lamens, A. Sur le processus non homogène de naissance et de mort à deux variables aléatoires. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 711–719.

The author treats a birth and death process involving two types of individual, "masculine" and "feminine". Each has its own time-dependent death rate. A feminine individual gives birth to a feminine individual with probability $p\lambda(t)dt+o(dt)$, to a masculine individual with probability $(1-p)\lambda(t)dt+o(dt)$ in time interval dt, whereas a masculine individual can only die. The forward and backward differential equations for the generating function of the population are found, and the first and second moments are thereby evaluated explicitly. When the death rates vanish identically, the generating functions are found explicitly. For the case of constant birth and death rates see also Goodman [Biometrics **9**(1953), 212–225; MR **14**, 1105] and Joshi [Publ. Inst. Statist. Univ. Paris **3** (1954), 153–177; MR **16**, 731].

J. L. Doob (Urbana, Ill.)

359:

***Пугачев, В. С. [Pugačev, V. S.] Теория случайных функций и ее применение к задачам автоматического управления. [Theory of random functions and its application to problems of automatic control.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 659 pp. (1 insert) 23.55 rubles.

This book is essentially a Russian counterpart of the recent text by Laning and Battin [Random processes in automatic control, McGraw-Hill, New York, 1956; MR **18**, 74]. (In the preface, the author notes that the Laning and Battin text appeared while the present text was in press, and that the former does not contain accounts of the results obtained by Soviet investigators.) The Russian book is very impressive, in terms of both the amount of information contained between its covers and the generally high quality of exposition. Much of the material is new, and a considerable part of it derives from the author's own papers on control systems and random processes.

Chapter 1: Introduction to probability theory. In contrast with the treatment of Laning and Battin, the author's presentation of the basic ideas is along orthodox lines, with no mention of the notion of sample space and related concepts. Chapters 2–6: Random variables. Distribution functions and densities. Normal and Poisson distributions. Characteristic functions. Transformation formulae. Law of large numbers and central limit theorems. Chapter 7: Entropy and information. The emphasis is on the continuous case, with the entropy of a variable ranging over the real line defined as $H[X]=-\int_{-\infty}^{\infty} p(x)\log[l_x p(x)]dx$, where $p(x)$ is the density and l_x is a fixed scale factor associated with X. By using appropriate scale factors, the entropy becomes invariant under 1-1 transformations, which is not true of the entropy defined in the conventional manner. Formulae for the entropies of normally distributed variables and discrete parameter Markov processes are derived and illustrated by concrete examples. Chapter 8: Random processes. The notion of a random process is introduced in an heuristic fashion. The emphasis is on manipulative techniques, with the derivations being, for the most part, purely formal. Detailed discussion of correlation functions and characteristic functionals of non-stationary processes. A brief discussion of the ergodic theorem. Chapters 8–9: Canonical expansions of random processes. A highly detailed treatment of the representation of scalar- and vector-valued random

processes in the form $X(t)=m_x(t)+\sum_\nu V_\nu x_\nu(t)$, where $m_x(t)=E[x(t)]$, the $x_\nu(t)$ are non-random functions and the V_ν are uncorrelated random variables. The well-known results of Karhunen [Ann. Acad. Sci. Fenn. Ser. A. I. no. 37 (1947); MR 9, 292] and others are extended in several directions, and many explicit formulae are derived. Chapter 10: Vector-valued processes. Correlation matrices and canonical resolutions. Chapter 11: Stationary processes. Definitions of stationarity and ergodicity. Canonical resolution of stationary processes. Chapters 12–13: Linear operations on random processes. Here, the author provides an unusually complete treatment of the transfer function of a time-varying system and its application to the analysis of slowly-varying systems. Chapters 14–15: Nonlinear operations on random processes. Approximate as well as exact representations of nonlinear operators are discussed in great detail. Considerable attention is given to the method of approximate linearization due to Kazakov [Avtomat. i Telemeh. **17** (1956), 385–409; MR **18**, 520] and, independently, Boonton [Proc. Symposium on Nonlinear Circuit Analysis, Polytech. Inst. Brooklyn, New York, 1953, pp. 369–391; MR **16**, 1036]. Tables of linearized parameters of various types of nonlinear devices are given in Appendix I. Chapter 16: Optimal systems. This is by far the longest chapter in the book (85 pages). It deals primarily with the design of both linear and nonlinear systems which are "best" in the sense of the mean-square error criterion. Very little space is given to Wiener's theory, the emphasis being on the less well-known methods employing canonical resolutions and various types of representations of nonlinear operators. The chapter closes with a brief but interesting discussion of the minimum-entropy criterion and its relation to the more conventional least-squares criterion. Chapter 17: Estimation of moments of random processes. An informative, but not very thorough, discussion of various methods of estimating the mean, variance, correlation function and spectral density function of a stationary process. *L. A. Zadeh* (New York, N.Y.)

STATISTICS

See also 335, 350, 419, 442, 771, 783, 787.

360:

Dwyer, Paul S. Generalizations of a Gaussian theorem. Ann. Math. Statist. **29** (1958), 106–117.

The theorem in question is the familiar least squares theorem of Gauss. Using the matrix derivative method [P. S. Dwyer and M. S. MacPhail, same Ann. **19** (1948), 517–534; MR **10**, 278], the author derives what appear to be the most general formulae, covering the case where the observable variables are correlated.

J. Neyman (Berkeley, Calif.)

361:

Krishna Iyer, P. V. A theorem on factorial moments and its applications. Ann. Math. Statist. **29** (1958), 254–261.

The theorem in question is a disguised form of a central result of the principle of inclusion and exclusion in a probability form; namely, that in consideration of n events with specified probabilities $P_{i_1\cdots i_k}$ for joint appearance of events i_1 to i_k, $k=1$ to n, and S_k the sum of such probabilities over all choices of k of the n events, the distribution by number of events appearing has a kth factorial moment which is equal to $k!S_k$. This result appears, e.g., in Fréchet, "Les probabilités associées à un système d'événements compatibles et dépendants, I" [Hermann, Paris, 1940; MR **3**, 168; p. 16]. The theorem is applied to various problems associated with Bernoulli trials. *J. Riordan* (New York, N.Y.)

362:

Siddiqui, M. M. Distribution of a serial correlation coefficient near the ends of the range. Ann. Math. Statist. **29** (1958), 852–861.

For the first autocorrelation coefficient r of N independent variables with normal distribution (0, 1), the expansion of $\text{Prob}(r>r_0)$ in powers of $1-r_0$ is shown to begin with $c(1-r_0)^{-(N-2)/2}$; $0.14<c<0.22$.

H. Wold (New York, N.Y.)

363:

Tukey, John W. Sums of random partitions of ranks. Ann. Math. Statist. **28** (1957), 987–992.

Suppose that the integers 1, 2, $\cdots$, N are randomly distributed among k distinguishable classes with equal probability and without restrictions. It is natural to denote the class sums by s_1, s_2, $\cdots$, s_k and largest of these by S. A generating function is obtained for the upper half of the range of S, namely $\frac{1}{k}N(N-1)\leqq S\leqq\frac{1}{2}N(N-1)$. For $k\leqq 6$, this is shown to provide the usual percentage points for N up to and beyond 10. Tables of 5% and 1% points are provided for $k=2$, 3, 6 and $N=1(1)10$.

For $k=2$, the distribution is that of Wilcoxon's paired sample test. This suggests the application of $k=6$ to the six possible orders of three responses. This is a possible procedure but the peculiarities of its power are such that its use is not recommended.

However, when three treatments with a natural order are examined in randomized blocks, a significance procedure can be based on the same distribution which is specifically sensitive to average responses in either exactly the same or exactly the opposite order as the treatments. 5% and 1% levels are given for $N=1(1)10$. The procedure may be promising. (From the author's summary) *L. A. Aroian* (Culver City, Calif.)

364:

Nicholson, W. L. On the distribution of 2×2 random normal determinants. Ann. Math. Statist. **29** (1958), 575–580.

In the quadratic form, $D=x_1x_4-x_2x_3$, let the independent random variables x_i be normally distributed with means μ_i, $i=1$, 2, 3, 4, and the common variance σ^2, which is then taken to be unity. The author gives the characteristic function of D, from which it appears that the distribution of D depends on two parameters, $\Lambda=\sum_1^4\mu_i^2$ and $\Delta=\mu_1\mu_4-\mu_2\mu_3$. The semi-invariants (cumulants) of D are readily written down; their form suggests that D is approximately normally distributed for large Λ; this is shown later in the paper. Inversion of the characteristic function gives the c.d.f. of D as a series with a remainder for which a readily computable bound is given which is good for $\Delta\geqq 0$ but less so for $\Delta<0$. It is observed that D is a special case of a more general quadratic form in the x_i, $i=1$, 2, 3, 4 which has the same c.d.f.; necessary and sufficient conditions on such a form for this result are given. Certain special cases greatly simplify the c.d.f.; its numerical approximation in general is discussed.

C. C. Craig (Ann Arbor, Mich.)

365:
Navrátil, Jan. Determination of the parameters of a compound distribution. Pokroky Mat. Fys. Astr. **3** (1958), 41–45. (Czech)

The problem of decomposition of a frequency distribution composed of two normal distributions with equal means is solved. The method employs absolute moments up to the third order. It is supposed that population moments can be replaced by corresponding sample moments with sufficient accuracy. A rather large sample size is thus required. *J. Janko* (Prague)

366:
Gupta, Shanti S.; and Sobel, Milton. On a statistic which arises in selection and ranking problems. Ann. Math. Statist. **28** (1957), 957–967.

Z, X, S are independent chance variables. Z has the distribution of the maximum of p independent normal variables, each with mean μ and variance σ^2. X has a normal distribution with mean μ and variance σ^2. $\nu S^2/\sigma^2$ has a chi-square distribution with ν degrees of freedom. Define $Y=(Z-X)/S$. Chance variables with the same distribution as Y arise in various selection problems. The authors give the moments of Y, approximations to the distribution function of Y, and tables of percentage points of the distribution of Y.

L. Weiss (Ithaca, N.Y.)

367:
Gjeddebaek, N. F. Contribution to the study of grouped observations. III. The distribution of estimates of the mean. Skand. Aktuarietidskr. **40** (1957), 20–25.

[For previous parts see Skand. Aktuarietidskr. **32** (1949), 135–159; **39** (1956), 154–159; MR **11**, 446; **19**, 992.] In this paper the author studies the effect of grouping observations on the efficiency of estimating the mean of a normal distribution, when sample sizes are small. If the distribution has variance 1 and the observations are grouped in cells of length h, two estimates are of interest. These are M, the "simple" mean based on assuming all observations in an interval are at the midpoint, and m, the maximum likelihood estimate (using the number of observations in each cell and not their actual values). Efficiencies are tabulated for sample sizes $N=1$ and $N=2$, and compared with $N=\infty$. For $h\leqq 2$, M and m have almost equal efficiencies which are high and seem to depend very little on N. The distributions of m and M are tabulated for $N=6$ and $h=2$.

H. Chernoff (Stanford, Calif.)

368:
Graybill, Franklin A. Determining sample size for a specified width confidence interval. Ann. Math. Statist. **29** (1958), 282–287.

The author shows how to construct confidence intervals whose lengths, with prescribed probability, do not exceed a given bound. The observations are taken in two stages, with the first sample being used solely to determine the proper size of the second sample from which the confidence intervals are determined. The method is applied to the problem of estimating the mean and variance of a normal distribution. *E. L. Lehmann* (Berkeley, Calif.)

369:
Tate, R. F.; and Goen, R. L. Minimum variance unbiased estimation for the truncated Poisson distribution. Ann. Math. Statist. **29** (1958), 755–765.

Let X_i, $i=1, 2, \cdots, n$, be a sample from a Poisson distribution truncated on the left. Define $t=\sum X_i$. The authors show that λ, the parameter of the Poisson distribution, possessses a minimum variance unbiased estimator given by t times the ratio of two generalized Stirling numbers of the second kind. If the Poisson distribution is truncated at $x=0$, tables are provided for this ratio for all combinations of n and t less than 49.

L. A. Aroian (Culver City, Calif.)

370:
Brudno, A. L. On a dispersion proof of the method of least squares. Mat. Sb. N.S. **43(85)** (1957), 37–48. (Russian)

The author proves that the least squares estimator of a linear function of the regression coefficients in the classical linear regression case has minimum variance in the class of all unbiased estimators. The classical theorem considers only linear estimators. *J. Wolfowitz* (Ithaca, N.Y.)

371:
Kesten, Harry. Accelerated stochastic approximation. Ann. Math. Statist. **29** (1958), 41–59.

Certain stochastic approximation procedures similar to that of A. Dvoretzky [Proc. Third Berkeley Symposium on Mathematical Statistics and Probability, vol. 1, Univ. of Calif. Press, Berkeley, 1956, pp. 39–55; MR **18**, 946] are considered, for which the magnitude of the nth step (i.e., $X_{n+1}-X_n$) depends on the number of changes in sign in (X_i-X_{i-1}) for $i=2, \cdots, n$. In Theorems 2 and 3, $X_{n+1}-X_n$ is of the form b_nZ_n, where Z_n is a random variable whose conditional expectation, given $X_1, \cdots, X_n$, has the opposite sign of $X_n-\theta$, and b_n is a positive real number. b_n depends in our processes on the changes in sign of X_i-X_{i-1} $(i\leqq n)$ in such a way that the more changes in sign before the nth step, the larger we make the correction on X_n to θ, when compared to the usual procedures [J. Kiefer and J. Wolfowitz, Ann. Math. Statist. **23** (1952), 462–466; MR **14**, 299; and H. Robbins and S. Monro, ibid. **22** (1951), 400–407; MR **13**, 144]. The author proves that the considered procedures converge with probability one. (From the author's summary.)

J. Wolfowitz (Ithaca, N.Y.)

372:
Gupta, Shanti S.; and Sobel, Milton. On selecting a subset which contains all populations better than a standard. Ann. Math. Statist. **29** (1958), 235–244.

Suppose we have $p+1$ populations, say $\pi_0, \pi_1, \cdots, \pi_p$, each population having the same type of distribution. The population π_0 is a "control" or "standard" population. A population is called "at least as good as" the control population if the specified parameter of interest of the population is at least as large as the corresponding parameter of the control population. A fixed quantity P^* $(0<P^*<1)$ is given. Observations will be taken on each of the populations $\pi_1, \cdots, \pi_p$, and on π_0 if the parameter of interest of this control population is unknown. The problem is to find a procedure which chooses a subset of the populations $\pi_1, \cdots, \pi_p$ such that the probability is at least P^* that the chosen subset contains all the populations which are at least as good as π_0. The following cases are discussed. (a) Populations normal, with a common known variance, the mean of the control population known, the mean being the parameter of interest. (b) Same as (a), except that the common variance is unknown. (c) Same as (a), except that the mean of the control population is unknown. (d) Same as (c), except that the common variance is also unknown. (e) Populations all chi-square, with known and possibly different degrees of freedom, the scale parameter of the control

population known, the scale parameter being of interest. (f) Same as (e), except that the scale parameter of the control population is unknown. (g) Populations binomial, the parameter of interest being the probability of success on a single trial, this parameter being known for the control population. (h) Same as (g), except that the parameter of the control population is unknown.

As an example of the type of solution the authors give, the rule for case (a) in the usual notation is: Retain in the selected subset those and only those populations π_i $(i=1, \cdots, p)$ for which $\bar{X}_i \geqq \mu_0 - d\sigma/\sqrt{n_i}$, where d is a constant chosen to give the desired probability P^*. The computation of d in this and the other cases is discussed.

L. Weiss (Ithaca, N.Y.)

373:

Block, H. D. Estimates of error for two modifications of the Robbins-Monro stochastic approximation process. Ann. Math. Statist. **28** (1957), 1003–1010.

The author studies two modifications of the Robbins-Monro stochastic approximation process [Ann. Math. Statist. **22** (1951), 400–407; MR **13**, 144]: $X_{n+1} = X_n - a_n(Y(X_n) - \alpha)$. The modifications are: (1) one takes several observations at the same X_n and employs the average of the observations in place of $Y(X_n)$; (2) the modification of (1) plus $a_n = a$ constant. It is assumed that the unknown regression lies between two straight lines with given slope, and that a given fixed number of observations is to be taken. For the two processes described above the author shows how to obtain a's optimal in a certain reasonable sense.

J. Wolfowitz (Ithaca, N.Y.)

374:

Olkin, Ingram; and Pratt, John W. Unbiased estimation of certain correlation coefficients. Ann. Math. Statist. **29** (1958), 201–211.

The traditional estimator for the correlation coefficient of a bivariate normal distribution is not unbiased, and it was not known whether unbiased estimators existed for this, and some related, parameters. The authors derive unbiased estimators for these parameters as functions of the traditional estimators, expressed in terms of hypergeometric functions. Each of the new estimators is shown to be best among all unbiased estimators for the parameter concerned. Asymptotic and other properties of the new estimators are explored. In particular, they are shown to have little or none of the undesirable behavior that is sometimes the price of unbiasedness. Tables are provided.

L. J. Savage (Rome)

375:

Plackett, R. L. Linear estimation from censored data. Ann. Math. Statist. **29** (1958), 131–142.

An asymptotic expansion of the expectation of the ith smallest value in a random sample of size n from a sufficiently regular distribution F, where $i = [n\alpha]$, $0 < \alpha < 1$, $n \to \infty$, is obtained using the logistic distribution, with bounds for the remainder term when F is normal. Let $y_1 \leqq y_2 \leqq \cdots \leqq y_n$ be an ordered random sample from a distribution with density $f((y-\mu)/\sigma)/\sigma$, and C_μ $[C_\sigma]$ the class of unbiased estimators of μ $[\sigma]$ which are linear functions of y_u, y_{u+1}, $\cdots$, y_v (u and v fixed). Under suitable assumptions, the asymptotic form of the minimum variance estimators in C_μ and C_σ is derived, and these estimators are shown to be asymptotically equivalent to the maximum likelihood estimators.

W. Hoeffding (Stanford, Calif.)

376:

Trybula, Stanisław. Some problems of simultaneous minimax estimation. Ann. Math. Statist. **29** (1958), 245–253.

In this paper, we give minimax estimates of the parameters of the multivariate hypergeometric distribution and of the multinomial distribution, and of some parameters of an unspecified distribution with known range. We use as loss a weighted linear combination of squared differences between the true and the estimated values of the parameters. (From the author's summary)

J. L. Hodges, Jr. (Berkeley, Calif.)

377:

Klepikov, N. P.; and Sokolov, S. N. Non-linear confluence analysis. Teor. Veroyatnost. i Primenen. **2** (1957), 473–475. (Russian. English summary)

A nonlinear confluence analysis problem is reduced to a regular regression problem by means of a series of approximations.

G. E. Noether (Boston, Mass.)

378:

Coulmy, Geneviève. Analyse et prédiction des marées. C. R. Acad. Sci. Paris **246** (1958), 1960–1962.

379:

Bašarin, G. P. The use of the chi-square criterion as a test for the independence of events. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 167–170. (Russian)

Let $A_1, A_2, \cdots, A_s$ be possible outcomes of an experiment. In N repetitions let m_{ij} be the number of times the pair A_iA_j occurs in sequence and m_i the number of times A_i occurs. Let $x_{ij} = (s^2/N)^{\frac{1}{2}}(m_{ij} - N/s^2)$. Under the assumption that the events are equally likely and independent, x_{ij} has asymptotically a normal distribution. However, $(s^2/N)\sum_{i,j=1}^{s}(m_{ij} - N/c^2)^2$ does not have a chi-square distribution because of the dependence of the m_{ij}. The author proves that

$$\frac{s^2}{N}\sum_{i,j=1}^{s}\left(m_{ij} - \frac{N}{s^2}\right)^2 - \frac{s}{N}\sum_{i=1}^{s}\left(m_i - \frac{N}{s}\right)^2$$

has a χ^2 distribution with $s^2 - s$ degrees of freedom. A similar result was proved by I. J. Good [Proc. Cambridge Philos. Soc. **49** (1953), 276–284; MR **15**, 727] for a slightly different statistic and for s prime. According to Good the above result is also implicit in the work of Bartlett [ibid. **47** (1951), 86–95; MR **12**, 512]. The proof presented here is very simple. Certain extensions are considered.

J. L. Snell (Hanover, N.H.)

380:

Suzuki, Yukio. Discrete decision problems. Ann. Inst. Statist. Math., Tokyo **9** (1958), 131–148.

The author discusses the maximum number of points at which optimal tests of hypotheses require randomization when the sample space consists of a finite number of points. {The import of Theorem 1 is obscured by the undefined term "equivalent".}

J. Wolfowitz (Ithaca, N.Y.)

381:

Weiss, Lionel. A test of fit for multivariate distributions. Ann. Math. Statist. **29** (1958), 595–599.

The hypothesis to be tested is that almost everywhere over a given region R the unknown density $f(y_1, \cdots, y_k)$ is equal to a given piecewise continuous function $g(y_1, \cdots, y_k) \geqq B > 0$ at every point of R, and $1 \geqq \int \cdots \int_R g(y_1, \cdots, y_k)\, dy_1 \cdots dy_k > 0$. The hypothesis says

nothing about $f(y_1, \cdots, y_k)$ outside the region R. The hypothesis is accepted if and only if certain inequalities involving the integral of $g(y_1, \cdots, y_k)$ over R are satisfied. There are other consistent tests for the hypothesis under discussion: the chi-square test and obvious extensions of the univariate Kolmogorov-Smirnov and von Mises tests. The author makes some conjectures about the asymptotic distributions of certain of the functions that enter into the test. *S. Kullback* (Washington, D.C.)

382:

Roy, S. N.; and Bargmann, R. E. Tests of multiple independence and the associated confidence bounds. Ann. Math. Statist. **29** (1958), 491–503.

A set of p variates with a joint normal distribution is divided into k subsets. The authors are concerned with a test for the mutual independence of the subsets, and with obtaining confidence bounds for certain regression coefficients. For the case $k=p$, the test proposed for testing $H: \sigma_{ij}=0$ $(i\neq j)$ is to reject H if $(1-r_i^2)<a$ for some $i=2, \cdots, p$, where r_i^2 is the multiple correlation coefficient of x_i on $(x_1, \cdots, x_{i-1})$. This is to be contrasted with the likelihood ratio test: reject H if $\prod_2^p (1-r_i^2)<c$.

I. Olkin (Stanford, Calif.)

383:

Fraser, D. A. S. Most powerful rank-type tests. Ann. Math. Statist. **28** (1957), 1040–1043.

To obtain a most powerful test against a given alternative among the tests which depend only on a function r of the observations (such as the vector of the ranks), it is sometimes convenient to express the probability density of r under the alternative in terms of an expectation taken with respect to the hypothesis distribution. Such an expression is stated in a general form and applied to the case of n independent random variables with a common probability density which is symmetric about 0 under the hypothesis and normal with positive mean under the alternative, with $r=(s_1, \cdots, s_n)$, where s_i denotes the sign of the observation which is the ith smallest in absolute value.

W. Hoeffding (Stanford, Calif.)

384:

Rosenstiehl, Pierre. Sur l'intégration de certaines fonctions caractéristiques simples ou composées. C. R. Acad. Sci. Paris **246** (1958), 2213–2215.

The author points out that if x is a random variable which is always positive and such that $E(x^{-\mu})$ exists, where p is a positive integer, then this quantity can be found by integrating the moment generating function for x, assumed to exist, p times over $(-\infty, 0)$. The procedure is extended to the multivariate case.

C. C. Craig (Ann Arbor, Mich.)

385:

Olkin, Ingram; and Pratt, John W. A multivariate Tchebycheff inequality. Ann. Math. Statist. **29** (1958), 226–234.

Let the random variables $y_1, \cdots, y_p$ have means 0 and a nonsingular covariance matrix, with $\sigma_i=(Ey_i^2)^{\frac{1}{2}}$, $\sigma_i\sigma_j\rho_{ij}=Ey_iy_j$. It is shown that

$$P(|y_i|\geq k_i\sigma_i \text{ for some } i)\leq \min_A \sum_{i,j=1}^{p} A_{ij}\rho_{ij}/(k_ik_j),$$

where $k_i>0$ and the minimum is taken with respect to all $p\times p$ matrices A such that $\sum_{i,j} A_{ij}x_ix_j\geq 1$ if $|x_i|=1$ for some i. The equality can be achieved whenever the right side is ≤ 1. For $p=2$, $k_1=k_2$ the inequality reduces to one by Berge [Biometrika **29** (1937), 405–406].

W. Hoeffding (Stanford, Calif.)

386:

Good, I. J. Saddle-point methods for the multinomial distribution. Ann. Math. Statist. **28** (1957), 861–881.

A theorem of Daniels [same Ann. **25** (1954), 631–650; MR **16**, 603] is slightly generalized in the case of discrete distributions. The principal generalization lies in the derivation of the third term of the asymptotic expansion. The two-variable and higher dimensional cases are also given. The applications are to the distributions of the maximum frequency and the χ^2-statistic in a multinomial when the class probabilities are equal. Some remarks on tests for the multinomial distribution and on properties of generating functions are included.

D. V. Lindley (Cambridge, England)

387:

Kaplan, E. L.; and Meier, Paul. Nonparametric estimation from incomplete observations. J. Amer. Statist. Assoc. **53** (1958), 457–481.

Assume (T_i, L_i), for $i=1, \cdots, N$, are independent and identically distributed with distribution function $F_{T_i, L_i}(x, y)=F(x)G(y)$. $t_i=\min(T_i, L_i)$ is observed. Let t_i' be the ith smallest of the t_i, that is, $t_1'\leq t_2'\leq \cdots \leq t_N'$. Then, the maximum likelihood estimate of $1-F(x)=\prod_r[(N-r)/(N-r+1)]$, where r runs over those indices such that $t_r'\leq x$ and t_r' is a T_i value. Moments, applications, and comparisons to other estimators are given for this estimator. In applications, $1-F(x)$ is the "survival function", T_i is the time of death (breakage, etc.) and L_i is the time of losing contact (withdrawal from testing, etc.).

I. R. Savage (Minneapolis, Minn.)

388:

Bennett, B. M. On the performance characteristic of certain methods of determining confidence limits. Sankhyā **18** (1957), 1–12.

The problem treated is that of set estimation of the quotient α of means of two normal variables. The paper contains excerpts from material developed in the 1940's. The main results published now were originally announced jointly by the author of the paper and the present reviewer [Ann. Math. Statist. **19** (1948), 116]. They represent the evaluation of the performance characteristics of two procedures found in the literature, occasionally believed to yield confidence intervals. The formulae now published, with some misprints, and also two numerical tables, show that these beliefs are erroneous. In fact, depending upon circumstances, the probability that the intervals yielded by the procedures studied will cover the true value of α may be arbitrarily close to zero. The deductions of the formulae are intermingled with manipulations having to do with the search of short unbiased confidence intervals for α. However, this latter problem is only vaguely mentioned. {The paper ends with expressions of thanks to Professor Evelyn Fix "for checking some of the tables presented in section 2." Actually, because of numerous errors, this checking amounted to a complete recalculation of the tables.} *J. Neyman* (Berkeley, Calif.)

389:

Mandel, John. A note on confidence intervals in regression problems. Ann. Math. Statist. **29** (1958), 903–907.

Suppose H is a joint confidence region of confidence coefficient $1-\alpha$ for the parameters $\theta_1, \cdots, \theta_k$, and $f_i(\theta_1, \cdots, \theta_k)$ $(i=1, \cdots, r)$ are r given functions. Define J as the region in $z_1, \cdots, z_r$ space consisting of all points $(z_1, \cdots, z_r)$ such that the simultaneous equations $f_i(\theta_1, \cdots, \theta_k)=z_i$ $(i=1, \cdots, r)$ are satisfied by at least

one point $(\theta_1, \cdots, \theta_k)$ in H. Then J is a joint confidence region for $f_1(\theta_1, \cdots, \theta_k), \cdots, f_r(\theta_1, \cdots, \theta_k)$ of confidence coefficient at least $1-\alpha$.

This is applied to the case where $Y_1, \cdots, Y_{N+p}$ are independent normal variates with a common variance and with $E\{Y_j\}=\beta_0+\beta_1 x_j$, where $x_1, \cdots, x_N$ are known constants, but $\beta_0, \beta_1, x_{N+1}, \cdots, x_{N+p}$ are unknown, and it is desired to construct a joint confidence region J for $x_{N+1}, \cdots, x_{N+p}$ by using H, the joint confidence ellipsoid for $\beta_0, \beta_1, E\{Y_{N+1}\}, \cdots, E\{Y_{N+p}\}$.

L. Weiss (Ithaca, N.Y.)

390:

Darling, D. A. The Kolmogorov-Smirnov, Cramér-von Mises tests. Ann. Math. Statist. **28** (1957), 823–838.

This is a systematically organized survey and exposition of nearly a hundred papers on the tests mentioned in the title, and others deriving from them.

J. L. Hodges, Jr. (Berkeley, Calif.)

391:

David, Herbert T. A three-sample Kolmogorov-Smirnov test. Ann. Math. Statist. **29** (1958), 842–851.

Three independent samples of size n each are drawn from three populations. The object is to test the hypothesis that the distribution functions of the populations are the same. Let $F_{1,n}(t)$, $F_{2,n}(t)$, $F_{3,n}(t)$ be the sample distribution functions of the three populations. The author considers the statistic

$$D_{3,n}=\text{Max}\ \{\sup_t(F_{2,n}(t)-F_{1,n}(t)),\ \sup_t(F_{3,n}(t)-F_{2,n}(t)),\ \sup_t(F_{1,n}(t)-F_{3,n}(t))\}$$

as the test statistic. The distribution functions of the populations are assumed to be continuous. Under the null hypothesis, the author shows that

$$\Pr\{D_{3,n}\geq l/n\}=3\sum_{i=1}^{[n/l]}\sum_{j\in J(i)}(\pm)(n!)^3/(n-il)!(n+jl)!(n+(i-j)l)!,$$

where the set $J(i)$ consists of the integers $(2-i, 3-i, 5-i, 6-i, 8-i, 9-i, \cdots, 2i)$, and where the $(\pm)$ sign indicates that for fixed i the successive terms in the finite series indexed by j have alternating signs beginning with $+$ for $j=2-i$. The result is obtained by reducing the problem to a random walk problem in the plane. The limiting distribution of $n^{\frac{1}{2}}D_{3,n}$ as $n\to\infty$ is then easily derived.

M. Rosenblatt (Bloomington, Ind.)

392:

Weiss, Lionel. Tests of fit in the presence of nuisance location and scale parameters. Ann. Math. Statist. **28** (1957), 1016–1020.

Certain functions of the sample spacings are shown to converge stochastically as the sample size increases. As an application, a test statistic based on sample spacings is given which provides a consistent test of fit against wide classes of alternatives for the following problem. Given a random sample from a population with cdf $F(x)$ and two constants u, v, $0\leq u\leq v\leq 1$; to test the hypothesis that in the interval $[F^{-1}(u), F^{-1}(v)]$, $F(x)$ equals some specified cdf $G(x)$ except for unspecified scale and location parameters. Only a conjecture about the asymptotic distribution of the test statistic is indicated.

G. E. Noether (Boston, Mass.)

393:

Birnbaum, Z. W.; and Pyke, Ronald. On some distributions related to the statistic D_n^+. Ann. Math. Statist. **29** (1958), 179–187.

Let $V_1<V_2<\cdots<V_n$ be the ordered values of n independent random variables, each uniformly distributed over the unit interval. Let $D_n^+=\max_t(i/n-V_t)=(i^*/n-V^*)$, when i^* is the index for which this maximum occurs and V^* the value of V_t for $i=i^*$. (These are well defined with probability 1). The joint and marginal distributions of V^* and i^* are obtained as well as some asymptotic results for i^*/n for large n. The methods are combinatorial.

M. Dwass (Evanston, Ill.)

394:

Birnbaum, Z. W.; and McCarty, R. C. A distribution-free upper confidence bound for $\Pr\{Y<X\}$, based on independent samples of X and Y. Ann. Math. Statist. **29** (1958), 558–562.

Let $p=\Pr(Y<X)$, where X and Y are independent chance variables. Let $\hat{p}=U/mn$, where U is the usual Wilcoxon-Mann-Whitney statistic based on m and n independent observations from X and Y, respectively. The paper provides a method for determining (conservative) numbers $M_{\epsilon,\alpha}$ and $N_{\epsilon,\alpha}$ such that $\Pr(p\leq\hat{p}+\epsilon)\geq 1-\alpha$, if $m\geq M_{\epsilon,\alpha}$ and $n\geq N_{\epsilon,\alpha}$. A table is provided which facilitates computations for $\alpha=.001, .005, .01, .05, .10$ and given sample ratios $\lambda=m/(m+n)=.1(.1).5$.

G. E. Noether (Boston, Mass.)

395:

Birnbaum, Z. W.; and Klose, Orval M. Bounds for the variance of the Mann-Whitney statistic. Ann. Math. Statist. **28** (1957), 933–945.

Let X, Y be independent random variables with continuous cumulative probability function and let $p=\Pr\{Y<X\}$. For the variance of the Mann-Whitney statistic U, upper and lower bounds are obtained in terms of p, for the case of any X and Y, as well as for the case of stochastically comparable X, Y. The results for the case of stochastic comparability are new, while the inequalities in the case of arbitrary X, Y have either been obtained by van Dantzig or are a consequence of other inequalities due to van Dantzig. (Author's summary)

Stochastic comparability means $P(X\leq t)\leq P(Y\leq t)$ for all t or $P(X\leq t)\geq P(Y\leq t)$ for all t. The paper of van Dantzig appeared in Nederl. Akad. Wetensch. Proc. Ser. A. **54**=Indag. Math. **13** (1951), 1–8 [MR **12**, 726].

M. Dwass (Evanston, Ill.)

396:

Savage, I. Richard. Contributions to the theory of rank order statistics — the "trend" case. Ann. Math. Statist. **28** (1957), 968–977.

Consider n independent random variables which, except for location parameters, all have the same distribution law. Assuming monotone-likelihood type distributions and a specified ordering for the location parameter, comparisons are made among the probabilities of various of the rank order regions. As the author says, "In spirit, the paper is a continuation of [I. R. Savage, Ann. Math. Statist. **27** (1956), 590–615; MR **18**, 243] and the techniques and terminology developed there will be used."

M. Dwass (Evanston, Ill.)

397:

Noether, Gottfried E. The efficiency of some distribution-free tests. Statistica Neerlandica **12** (1958), 63–73. (Dutch summary)

An expository summary of results on the asymptotic

local efficiencies of nonparametric tests relative to their parametric competitors.

J. L. Hodges, Jr. (Berkeley, Calif.)

398:

***Kitagawa, Tosio; and Mitome, Michiwo. Tables for the design of factorial experiments.** Revised ed. Dover Publications, Inc., New York, 1958. vii+253 pp. $8.00

This useful volume gives a comprehensive listing of the following types of experimental designs: Latin squares and cubes, factorial designs, fractional replications in factorial designs, factorial designs with split plot confounding, factorial designs confounded in quasi-Latin squares, lattice designs, balanced incomplete block designs and Youden squares. The listing seems to comprise all designs of these types that have proved useful in practice. Several balanced incomplete block designs with more than ten replications are listed.

H. B. Mann (Columbus, Ohio)

399:

Searle, S. R. Sampling variances of estimates of components of variance. Ann. Math. Statist. **29** (1958), 167–178.

Using matrix methods, the author calculates the sampling variances of estimates of components of variance from data in a 2-way classification having unequal subclass numbers.

P. Whittle (Wellington)

400:

Zelen, Marvin. The use of group divisible designs for confounded asymmetrical factorial arrangements. Ann. Math. Statist. **29** (1958), 22–40.

The various designs used for treatment comparisons may be used for factorial experiments if one considers the treatment combinations as treatments and arranges them in blocks according to the design used. The author, in particular, considers factorial experiments arranged in this way in group divisible designs. For these arrangements, the author derives in detail the intra- and interblock analysis and tests of significance for main effects and higher order interactions. *H. B. Mann* (Columbus, Ohio)

401:

Rao, V. R. A note on balanced designs. Ann. Math. Statist. **29** (1958), 290–294.

Consider a design whose incidence matrix is $N=(n_{ij})$, $i=1, 2, \cdots, v$; $j=1, 2, \cdots, b$; here v and b denote, as usual, the number of varieties and the number of blocks. The ith treatment is replicated r_i times and the jth block contains k_j varieties. The design is said to be balanced if every elementary contrast is estimated with the same variance. The author proves that a necessary and sufficient condition for a design to be balanced is that the matrix C have $v-1$ equal latent roots other than zero, where $C=\text{diag}(r_1, \cdots, r_v)-N\,\text{diag}(k_1^{-1}, \cdots, k_b^{-1})N'$. It follows as a corollary that if all the k_j are equal, then all the r_i are equal, and the design is a BIBD, if it exists; however, the r_i may all be equal without requiring that all the k_j be equal.

R. G. Stanton (Waterloo, Ont.)

402:

Blasbalg, H. Transformation of the fundamental relationships in sequential analysis. Ann. Math. Statist. **28** (1957), 1024–1028.

A sequential test of one hypothesis against another with prescribed probabilities of the two kinds of error is, in general, a function of four parameters, namely, the two hypotheses and the two probabilities, α and β. The author exhibits a family of one-dimensional distributions $P(x, \theta)$ depending on a real parameter θ, which is such that the sequential probability ratio test of $H_0:\theta\leqq\theta_0$ against $H_1:\theta\geqq\theta_1$ $(\theta_1>\theta_0)$ depends only on three functions of α, β, θ_0 and θ_1.

S. G. Ghurye (Chicago, Ill.)

403:

Savage, Leonard J. When different pairs of hypotheses have the same family of likelihood-ratio test regions. Ann. Math. Statist. **28** (1957), 1028–1032.

This paper is motivated by the following problem: Given two pairs of probability distributions (F, G) and (F', G'), when is the family of sequential probability ratio test regions for discriminating between F and G the same as that for (F', G')? This question is treated in general for pairs of probability measures on an abstract space and it is shown that a necessary and sufficient condition is that the likelihood ratio of F with respect to G be a constant multiple of that of F' with respect to G'[mod$(F+G+F'+G')$]. This result is used to show that "the logarithmic densities of a family that is, so to speak, degenerate with respect to sequential tests can with more than sufficient generality be represented by" $h_0(x)+\eta_1(\theta)h_1(x)+\eta_2(\theta)h_2(x)$. The reasoning in this latter part is heuristic. The distributions discovered by Blasblag [see article reviewed above] are, naturally, contained in this family.

S. G. Ghurye (Chicago, Ill.)

404:

Quensel, Carl-Erik. A contribution to the theory of correlation and regression in non-random samples. Lunds Univ. Årsskrift. N. F. Avd. 2. **54**=Kungl. Fysiogr. Sällsk. Handl. N. F. **69** (1958), no. 7, 31 pp.

To assess bias due to selection, the author studies a two-variate normal distribution, specifies for one variate a normal selection mechanism with parameters that can be estimated from the sample, and evaluates the large-sample distortion in the regression statistics. Extension to three variates and applications to differences in school results arising from selection between education lines are included.

H. Wold (New York, N.Y.)

405:

Striebel, Charlotte T. On the efficiency of estimates of trend in the Ornstein Uhlenbeck process. Ann. Math. Statist. **29** (1958), 192–200.

Let x_t be a normal process and $y_t=x_t+f_t$, where $f(t)=K_1\phi_1(t)+K_2\phi_2(t)+\cdots+K_s\phi_s(t)$ and $\phi_1(t), \cdots, \phi_s(t)$ are given functions. It is standard procedure to estimate the K_i by the maximum likelihood method. However, if the covariance function $C(u, v)$ of x_t contains unknown parameters, this often leads to equations which are too complicated to solve. In this case, one may try to use another estimate $f(t)$, and compare it with the maximum likelihood estimate $\hat{f}(t)$ by computing the efficiency

$$e(T)=E\left(\int_0^T (\hat{f}(t)-f(t))^2dt\right)\Big/E\left(\int_0^T (\hat{f}(t)-f(t))^2dt\right).$$

The limit $\lim_{T\to\infty} e(T)=e$ is called the asymptotic efficiency of $f(T)$. The least square estimate is easily computed and independent of $C(u, v)$. The author considers two classes of processes y_t and derives in both cases necessary and sufficient conditions under which the asymptotic efficiency $\bar{e}$ of the least square estimate is unity. Applying her results to the Ornstein Uhlenbeck process, the author first improves on a result by the reviewer and P. B. Moranda [Sankhyā **13** (1954), 351–358; MR **16**, 154] and shows further that $\bar{e}<1$ if the $\phi_i(t)$ are exponential func-

tions. In this case, she proposes another estimate and shows that it is superior to the least square estimate.

H. B. Mann (Columbus, Ohio)

406:

Wierzbicki, Witold. Probabilistic and semi-probabilistic method for the investigation of structure safety. Arch. Mech. Stos. **9** (1957), 685–694. (Polish and Russian summaries)

Probability of non-catastrophic failure of a structure is determined in terms of (a) probabilities of error due to simplified (i.e. non-accurate) calculations, (b) variations in elastic constants, (c) variations in material properties. No specific calculations are made.

J. Heyman (Cambridge, England)

NUMERICAL METHODS

See also 25, 130, 131, 234, 261, 325, 458, 459, 474, 555, 558, 777.

407:

***Scarborough, James B. Numerical mathematical analysis. 4th ed.** The Johns Hopkins Press, Baltimore, Md.; Oxford University Press, London; 1958. xxi+576 pp. $6.00.

Previous editions have been reviewed in MR **12**, 537 and MR **17**, 789. The present edition incorporates the following changes: an article on the convergence of the Newton-Raphson method has been added; Brodetsky and Smeal's improvement of Graeffe's method is explained in detail and somewhat simplified; an article on improving the accuracy of complex roots found by the Graeffe method has also been added, as has a section on smoothing experimental data.

408:

Engeler, Erwin. Über die Monte-Carlo-Methode. Mitt. Verein. Schweiz. Versich.-Math. **58** (1958), 67–76.

Elementary expository article.

W. Wasow (Madison, Wis.)

409:

Wendel, J. G. Groups and conditional Monte Carlo. Ann. Math. Statist. **28** (1957), 1048–1052.

H. F. Trotter and John W. Tukey, [Symposium on Monte Carlo methods; Wiley, New York; Chapman and Hall, London; pp. 64–79; MR **18**, 152] proposed a procedure for modifying a random variable so that a conditional expectation could be treated as an unconditional expectation of a properly weighted random variable in order to have a more efficient Monte Carlo method. That is, given a random variable X with density f, a Borel measurable function φ, the problem is to express the expectation $E_f(\varphi(x)|A(X)=x_0)$ as an unconditional expectation $E_g(\varphi(X^*)w^*(X))$, where X is sampled from density g, X^* is a suitable modification of X, and w^* is an appropriate weight. The present paper shows, under rather general conditions, that it is possible to find a weight function w^* and an appropriate modification X^* of X so that

$$E_g(\varphi(X^*)w^*(X))=E_f(\varphi(X)|\text{conditions on } X)$$

identically is φ.

In the review of the Trotter and Tukey paper mentioned above, this reviewer drew attention to the need to take into account the variance of this type of modified Monte Carlo procedures. Using a very simple (binomial type) example the author presents a useful discussion of this problem, showing explicitly how the variance is influenced by the specific set of weights selected.

M. Muller (Princeton, N.J.)

410:

Woollett, Enid R. Subtabulation with special reference to a high-speed computer. Quart. J. Mech. Appl. Math. **11** (1958), 185–195.

"The method of L. J. Comrie [J. Royal Stat. Soc. Suppl. III, **2** (1936), 87–113] and E. C. Bower [Lick Obs. Bull. **467** (1935), 65–74] for subtabulating a function by use of bridging differences is generalized to cover all subintervals where the procedure gives exact pivotal values. Two other methods are given; one uses leading differences and the other corrections to all odd orders of differences. The merits of the three methods for use with an automatic computer are discussed. It is shown that any desired accuracy can be obtained even when the pivotal values are not built up exactly. The method can be applied to all subtabulation ratios including those that are non-integral or even irrational. Various worked examples are given." (Author's summary.)

John Todd (Pasadena, Calif.)

411:

Minnick, Robert C. Tshebysheff approximations for power series. J. Assoc. Comput. Mach. **4** (1957), 487–504.

L'auteur propose un perfectionnement des méthodes jusqu'ici employées dans l'approximation d'une fonction par les polynomes de Tschebyscheff, et son application au calcul numérique des valeurs de la fonction. Ces méthodes essentiellement dues à Lanczos [cf. J. Math. Phys. **17** (1938), 123–199] et à Hastings [Approximation for digital computers, Princeton Univ. Press, 1955; MR **16**, 963] se trouvent ici améliorées par un procédé qui, sans augmenter la valeur absolue de l'écart dans chaque intervalle, diminue le nombre de termes dans les séries utilisées.

Une application est donnée au calcul numérique de e^x et de $J_0(x)$ pour $-1\leqq x\leqq+1$. *R. Campbell* (Caen)

412:

Shenitzer, A. Chebyshev approximation of a continuous function by a class of functions. J. Assoc. Comput. Mach. **4** (1957), 30–35.

Résultats obtenus, avec une machine à calculer IBM 650, pour la méthode d'approximation polynomiale de Novodvorskii-Pinsker [Uspehi Mat. Nauk (N.S.) **6**, no. 6(46) (1951), 174–181; MR **13**, 728].

L. Schwartz (Paris)

413:

Šura-Bura, M. R. Approximation of functions of many variables by means of functions each of which depends on one variable. Vyčisl. Mat. **2** (1957), 3–19. (Russian)

L'auteur cherche le minimum de l'intégrale

$$(*)\quad \int_0^1\int_0^1\Big[F(x,y)-\varphi_0(x)-\psi_0(y)-\sum_{k=1}^{n}\mu_k\varphi_k(x)\psi_k(y)\Big]^2dxdy,$$

où $F(x, y)$ est la fonction donnée, et $\varphi_i(x)$, $\psi_i(y)$ normées par

$$\int_0^1\varphi_i dx=\int_0^1\psi_i dy=0,\quad \int_0^1\varphi_i^2dx=\int_0^1\psi_i^2dy=1.$$

Soit

$$F_1(x,y)=F(x,y) - \int_0^1 F(x,\eta)d\eta - \int_0^1 F(\xi,y)dy - \int_0^1\int_0^1 F(\xi,\eta)d\xi d\eta$$

le noyau de l'opérateur $A\varphi=\int_0^1 F_1(x,y)\varphi(y)dy$, alors les fonctions φ_i et ψ_i minimisant (*) sont les paires de fonctions liées par les relations

$$\varphi_k=\lambda_k A\psi_k, \qquad \psi_k=\lambda_k A^*\varphi_k,$$
$$\varphi_k=\lambda_k{}^2AA^*\varphi_k, \quad \psi_k=\lambda_k{}^2A^*A\psi_k,$$

correspondants aux n valeurs propres $\lambda_1{}^2\leq\cdots\leq\lambda_n{}^2$ des opérateurs autoadjoints A^*A et AA^*, $\mu_k=\lambda_k{}^{-1}$. On applique ce procédé à l'approximation numérique de fonction de Bessel $J_p(x)=F(p,x)$, $0\leq p\leq 1$, $1\leq x\leq 2$.

M. Tomić (Beograd)

414:

Holladay, John C. A smoothest curve approximation. Math. Tables Aids Comput. **11** (1957), 233–243.

Attention is given in this paper to requiring an approximating curve to be smooth by minimizing the integral of the square of the second derivative of the approximating function. The following theorem is proved.

Theorem. Let $t_0<t_1\cdots<t_n$. For each $i=0, \cdots, n-1$, let f be a cubic polynomial on the interval $[t_i, t_{i+1}]$. Also let $d^2f(t_0)/dt^2=f''(t_0)=0=f''(t_n)$. For $i=1, \cdots, n-1$, let f, f', f'' be continuous at t_i. Let g be any admissible function on $[t_0, t_n]$ such that $g(t_i)=f(t_i)$ for $i=0, \cdots, n$. Then $\int_{t_0}^{t_n}[g''(t)]^2dt>\int_{t_0}^{t_n}[f''(t)]^2dt$ if and only if $g\neq f$.

From the proof it seems that an admissible function is one with continuous second derivative. Applications are made to obtain quadrature formulas and other theorems concerning polynomial quadrature are given.

In a letter to the reviewer, the author writes: "The part at the end of the proof of Theorem 7, starting with 'which using the binomial expansion, equals,' should be at the end of the proof of Theorem 9. The part at the end of the proof of Theorem 8, starting with, and

$$D(2n+1)=2[y^{2n+1}-(-y)^{-(2n+1)}]'$$

should be at the end of the proof of Theorem 11. The part at the end of the proof of Theorem 11, starting with 'to get

$$w(m, n)-1='$$

should be at the end of the proof of Theorem 7."

With these changes, the proof of Theorem 11 seems not to be complete. *P. C. Hammer* (Madison, Wis.)

415:

Salzer, Herbert E. Note on multivariate interpolation for unequally spaced arguments, with an application to double summation. J. Soc. Indust. Appl. Math. **5** (1957), 254–262.

L'auteur établit des formules pour l'interpolation d'une fonction de m variables, par un polynôme de degré total n sous la forme de Lagrange, et lorsque les points utilisés ont pour coordonnées $(x_i, y_j, \cdots, z_k)$, $0\leq i\leq n$, $0\leq j\leq n$, $0\leq k\leq n$, $0\leq i+j+\cdots+k\leq n$.

Il donne différentes expressions des coefficients intervenant dans la formule de Lagrange en particulier au moyen de coefficients de Lagrange relatifs à une seule variable.

Ces formules sont appliquées à la sommation de séries doubles en supposant que la somme $S_{p,q}$ est un polynôme de degré convenable en p^{-1}, q^{-1}.

La méthode est appliquée à la détermination de la constante d'Euler et à trois exemples simples donnant comme limite 0 ou 1. *J. Kuntzmann* (Grenoble)

416:

Hahn, E. Berechnung der zu einer vorgegebenen Differenzenfunktion beliebig hoher Ordnung nach der Methode der kleinsten Quadrate gehörigen Stammfunktion. Z. Angew. Math. Mech. **38** (1958), 120–130. (English, French and Russian summaries)

The n values $f_k=f(x_k)$ at the points x_k of the interval $x_1\leq x_k\leq x_n$ are taken as variable, while the values f_k, $k<1$ and $k>n$, outside this interval are given. The $n+p$ central differences of order p, $(\delta^p f)_j$ defined by

$$(\delta^p f)_j=(-1)^p\sum_{k=1}^{n}(-1)^{k+\frac{1}{2}p-j}\binom{p}{k+\frac{1}{2}p-j}f_j,$$

for $j=1-p/2, 2-p/2, \cdots, n+p/2$, which depend on f_k, $k=1, 2, \cdots, n$, are denoted by y_j. A recurrence formula is derived which determines the values of f_k, $k=1, 2, \cdots, n$, in such a way that the expression

$$\sum_{j=1-\frac{1}{2}p}^{n+\frac{1}{2}p}\{(\delta^p f)_j-y_j\}^2$$

becomes a minimum. *D. Moskovitz* (Pittsburgh, Pa.)

417:

Goldstein, Allen A.; Levine, Norman; and Hereshoff, James B. On the "best" and "least qth" approximation of an overdetermined system of linear equations. J. Assoc. Comput. Mach. **4** (1957), 341–347.

For $q>1$ (preferably a power of 2) and linear forms L_i, $\sum|L_i|^q$ is minimized by either of two gradient procedures. For large q the solution approximates that of $\min\max_i|L_i|$. Numerical examples, including minimax approximation by polynomials in one or two variables. The method is also used effectively in alternation with de la Vallée Poussin's method of subset selection.

T. S. Motzkin (Los Angeles, Calif.)

418:

Dwyer, Paul S.; and Galler, Bernard A. The method of reduced matrices for a general transportation problem. J. Assoc. Comput. Mach. **4** (1957), 308–313.

This paper is an abstract of an unpublished paper. It does not contain full details of the methods it describes.

Reduced matrices are those obtained when constants are subtracted from each row (column, etc) to make the minimum of each row (column, etc) zero. The method proceeds in three stages. Starting with a reduced matrix, transformations of it are determined which after a few iterations give a solution satisfying the equations of the problem but perhaps not all the inequalities. Further transformations are determined in order to satisfy the inequalities too. The optimum solution has now been reached but it may not have integral elements. An integral solution is then obtained from the fractional solution and some necessary conditions from the reduction process.

A method is given of finding a good initial feasible solution which may also be useful in the simplex procedure. Weighted deviates of the costs are calculated. A solution is constructed by using the minimum deviate to capacity, and then the next minimum and so on.

Both these procedures have been programmed for automatic computers.

M. Fieldhouse (Cambridge, England)

419:

Nordbotten, Svein. On errors and optimal allocation in a census. Skand. Aktuarietidskr. **40** (1957), 1–10.

In a census, N characteristics of a population are measured by performing K processes (e.g., planning, enumeration, editing). Let t_k denote a measure of the

effort devoted to the kth process. Assume that the total cost of the census is a linear function of the t_k, viz., $c=c_0+\sum c_k t_k$, $c_k>0$, $k=1, \cdots, K$, and that the value of the ϵ-percentile of the error distribution of the ith characteristic is similarly affected by effort, i.e., $E_i=A_i+\sum B_{ik}t_k$, $B_{ik}\leq 0$, $i=1, \cdots, N$, $k=1, \cdots, K$. Then the problem of allocating a census budget optimally can be stated: Choose $t_k\geq 0$, $k=1, \cdots, K$, so as to minimize c, subject to $E_i\leq$ preassigned constants, $i=1, \cdots, N$. This formulation reduces the census budget allocation to a standard linear programming problem. For more realism, positive lower limits can be imposed on the t_k and the other strict assumptions can be relaxed somewhat.

R. Dorfman (Cambridge, Mass.)

420:

Bonner, Raymond E.; Kosowsky, Lester H.; and Ordung, Philip F. Functional characteristics of a node determinant. J. Franklin Inst. **265** (1958), 395–406.

The authors consider the expansion of the nth order network determinant (a_{ij}) having

$$a_{ij}=\begin{cases}-y_{ij}, & i\neq j;\\ \sum_{x=1}^{n} y_{ix}, & i=j.\end{cases}$$

A modification of the Laplace expansion is utilised for the removal of the negative terms. It is shown that the formal expansion has $\binom{n-1}{n+1}$ terms, each with a positive sign. [See Ramaswami Aiyar, J. Indian Math. Soc. **18** (1930), 189–207.] *H. Gupta* (Hoshiarpur)

421:

Khan, N. A. The characteristic roots of the product of two matrices. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 348–356.

The problem considered is rather special since the matrices A and B are commuting and, moreover, $A+A^*$, $(A-A^*)/i$, $B+B^*$, and $(B-B^*)/i$ are assumed positive semi-definite. Four theorems are presented bounding the real and imaginary parts of the characteristic values of AB in terms of the largest and smallest characteristic values of the above Hermitian components.

A. S. Householder (Oak Ridge, Tenn.)

422:

Lanczos, C. Iterative solution of large-scale linear systems. J. Soc. Indust. Appl. Math. **6** (1958), 91–109.

The present paper is concerned with the solution of a system of algebraic equations $Ax=b$ in which the elements of A and b are known only to a certain degree of accuracy. It is assumed further that the matrix A is ill-conditioned. The problem at hand is to obtain an approximate solution that depends principally upon the larger eigenvalues and in a certain sense is the significant part of the solution. The method used is an iterative method based upon the properties of Chebyschev polynomials. A detailed analysis of the method is given, with practical suggestions of how to apply the method effectively. The method is undoubtedly a practical one for a large class of problems arising in application. *M. Hestenes* (Los Angeles, Calif.)

423:

Mayer, Daniel; and Schmidtmayer, Josef. Representation of inverse matrices by convergent geometrical series. Apl. Mat. **2** (1957), 24–37. (Czech. Russian and English summaries)

An expository paper discussing the method of computing A^{-1} based upon a representation of A as a sum of the diagonal matrix D of its diagonal elements and the remaining matrix $-Q$. If $S=D^{-1}Q$, then $A^{-1}=(I+S+S+\cdots)D^{-1}$. Notes on convergence and two numerical examples, one in connection with an electrical network, are given. *H. Schwerdtfeger* (Montreal, P.Q.)

424:

Hecht, Josef. A note on the solution of systems of linear algebraic equations. Apl. Mat. **3** (1958), 233–237. (Czech. Russian and English summaries)

Instead of representing the $n\times n$-matrix A as product of a subtriangular and a supertriangular matrix, the author proposes a representation of this kind for the $2n\times 2n$-matrix $\begin{pmatrix}A & 0\\ E & -E\end{pmatrix}=\begin{pmatrix}L & 0\\ M^{-1} & -E\end{pmatrix}\begin{pmatrix}M & 0\\ 0 & E\end{pmatrix}$, at the same time replacing the system of linear equations $Ax=r$ by $\begin{pmatrix}A & 0\\ E & -E\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}r\\ 0\end{pmatrix}$. No evident advantage seems to result from this procedure. There is also a description of a variant of this method and two numerical tables, given without comment. *H. Schwerdtfeger* (Montreal, P.Q.)

425:

Dimsdale, B. The non-convergence of a characteristic root method. J. Soc. Indust. Appl. Math. **6** (1958), 23–25.

Greenstadt [Math. Tables Aids Comput. **9** (1955), 47–52; MR **17**, 411] suggested an iteration procedure to triangularize arbitrary matrices. The present paper gives a class of matrices for which this procedure does not converge. *P. Rabinowitz* (Rehovoth)

426:

Wittmeyer, H. A new method for developing simple formulae for the eigenvalues of linear ordinary self-adjoint differential equations. J. Soc. Indust. Appl. Math. **6** (1958), 111–143.

The method outlined considers an equivalent variational problem, transforms this so that the corresponding differential equation is in normal (Liouville) form, replaces the functions in the transformed variational problem by constants, and hence solves a differential equation with constant coefficients. The first eigenvalue of this equation approximates closely to that of the original problem, and its higher eigenvalues have the correct asymptotic form. Formulae are given for the best choice of the constants, and results are obtained for the torsional and flexural oscillations of a beam, in close agreement with the known solutions. An outline is given of a generalisation of the method for more complicated cases involving differential equations of higher order.

L. Fox (Teddington)

427:

***Diaz, J. B. Upper and lower bounds for eigenvalues.** Inst. for Fluid Dynamics and Appl. Math., Univ. of Maryland, College Park, Md., 1956. ii+47 pp.

428:

Lehmer, Emma; and Vandiver, H. S. On the computation of the number of solutions of certain trinomial congruences. J. Assoc. Comput. Mach. **4** (1957), 505–510.

The authors consider the congruences $1+ax^s\equiv by^l \pmod p$, where p is an odd prime, $cl+1=p$ and $xy\not\equiv 0 \pmod p$. An equivalent form is $1+g^{i+cr}\equiv g^{j+ls} \pmod p$, where g is a primitive root of p, $0\leq i\leq c-1$, $0\leq j\leq l-1$, $0\leq r\leq l-1$ and $0\leq s\leq c-1$. Sets of congruences with given i and varying j are classified according to the numbers of solutions occurring for different j. Tables (calculated by

the electronic computer SWAC) of the number of i in each class are given for $l=5$, $p\leqq 971$ and $l=7$, $p\leqq 1009$. Some statistical abnormalities in the distributions are pointed out but no explanation is found. A relation exists between these congruences and the Fermat problem.

C. B. Haselgrove (Manchester)

429:

Christensen, Bent. Discussion and solution of the cubic equation. A new method for use in practical calculations. Acta Polytech. **238** (1957), iii+19 pp.

430:

Peirce, William H. Numerical integration over the spherical shell. Math. Tables Aids Comput. **11** (1957), 244–249.

The author gives numerical integration formulas for spherical shells which are of arbitrarily high polynomial accuracy. The formulas are of product type and are derived using this spherical coordinate representation. The evaluation points depend on zeros of Legendre polynomials and zeros of another class of polynomials. A table of these latter is given for a few values of the inner radius of the shell. This paper makes numerical quadrature over a spherical shell and its affine transforms a feasible operation. *P. C. Hammer* (Madison, Wis.)

431:

Stroud, A. H. Remarks on the disposition of points in numerical integration formulas. Math. Tables Aids Comput. **11** (1957), 257–261.

Formulas for numerical estimations exact for polynomial integrand functions of degree 2 and 3 for symmetrical regions in E_n are given. These formulas involve $n+1$ and $2n$ evaluation points respectively. It is shown that rotations of these sets of evaluation points do not affect the polynomial degree of accuracy and, as one application, a 3rd degree formula given by Tyler for the n-cube is rotated so that the evaluation points fall inside the region. This paper displays some of the distinctive differences between the problem of numerical integration for one variable and that for several.

P. C. Hammer (Madison, Wis.)

432:

Albrecht, J.; und Collatz, L. Zur numerischen Auswertung mehrdimensionaler Integrale. Z. Angew. Math. Mech. **38** (1958), 1–15. (English, French and Russian summaries)

The authors give formulas for the numerical evaluation of two- and three-dimensional integrals. Taylor series about the centroid of the region are used to develop certain remainder terms. For the planar case, the circle, the equilateral triangle, the square, and the regular hexagon are singled out for certain formulas exact for polynomials up through at most fifth degree. Formulas of this sort given by Radon, Mikeladze, Willers, Runge, Tyler, Sadowsky, and Hammer, Marlowe and Stroud are listed. Also, formulas for regular polytopes in three-dimensional space are given. {While this paper contains many specific formulas, it leaves out many which had been developed. Among these are the 27-point formulas of J. Clerk-Maxwell for the cube, the four and five-point formulas of Hammer, Marlowe and Stroud for the tetrahedron [Math. Tables Aids Comput. **10** (1956), 130–137, 137–139; MR **19**, 177], and the 12-point formula of Tyler for the square. The cartesian product principle stated by Hammer and Wymore [ibid. **11** (1957), 59–67; MR **19**, 323] makes it simple to calculate formulas for low-dimensional cubes from formulas for the line segment. This observation is not used by the authors. More recently published works of Peirce give numerical formulas for the circular annulus [J. Soc. Indust. Appl. Math. **5** (1957), 66–73; MR **19**, 771] and the spherical shell [#430 above], which are of any specified degree. Using these and the conical product principle given by Hammer, Marlowe and Stroud [loc. cit.] and principles of symmetry stated by Hammer and Wymore [loc. cit.] in conjunction with explicit formulas in this paper, one begins to have a fair range of useful regions to which formulas may be applied. The results mentioned in this paragraph were not available to the authors. *P. C. Hammer* (Madison, Wis.)

433:

Coulmy, Geneviève. Opérations sur les courbes expérimentales. C. R. Acad. Sci. Paris **246** (1958), 1799–1800.

The author proposes using the formulas $y_0'=[8(y_1-y_{-1})-(y_2+y_{-2})]/12h$ and $y_0''=[16(y_1+y_{-1})-(y_2+y_{-2})-30y_0]/12h^2$ to estimate first and second derivatives from equally spaced data y_i. For numerical integration she recommends the formula:

$$y_{0,n}=h\Big[\frac{13}{16}(y_0+y_n)+\frac{7}{6}(y_1+y_{n-1})+y_2+y_{n-2}+\frac{35}{36}(y_3+y_{n-3})+\sum_{q=4}^{n-4}y_q\Big].$$

The latter she suggests as applicable in the problem of tides. *P. C. Hammer* (Madison, Wis.)

434:

Boley, Bruno A. A method for the numerical evaluation of certain infinite integrals. Math. Tables Aids Comput. **11** (1957), 261–264.

This note discusses the numerical evaluation of an integral of the form

$$(1)\qquad I(a)=\int_a^\infty f(x)\sin x\,dx;$$

$f(x)$ descends to zero as $x\to\infty$. Assuming also that $f^{(k)}(x)$ descends to zero as $x\to\infty$ for all k, the integral is rewritten as

$$(2)\quad I(a)=\sum_{i=0}^{\infty}f^{(i)}(a_{i+1})\cos[a_{i+1}+i(\pi/2)]+\sum_{i=0}^{\infty}\int_{a_i}^{a_{i+1}}f^{(i)}(x)\sin[x+i(\pi/2)]\,dx,$$

where one has $a_0=a$ and $a_{i+1}\geqq a_i$.

It is then established that there exists a sequence $a_1\leqq a_2\cdots\leqq a_n<\cdots$, such that the series on the right of (2) both converge and their sum is $I(a)$. An example is given using $f(x)=x^{-k}$, where $k>0$. The author states that the method may be used in generalization for other types of integrals including those in which the integrand is not oscillating. *P. C. Hammer* (Madison, Wis.)

435:

Muraviev, P. A. Integration of a system of linear differential equations with constant coefficients and retarded arguments. Ukrain. Mat. Ž. **9** (1957), 432–441. (Russian. English summary)

The system of linear first order differential equations

$$x'(t)=\sum_{j=0}^{r}A_jx(t-\tau_j)+F(t)$$

($\tau_0=0<\tau_1<\tau_2<\cdots<\tau_r$ are given constants, $x(t)$ is a

vector with n components, $F(t)$ is a given vector and continuous for all t and the A_k are (n, n) matrices with given constant elements) is considered together with the initial condition $x(0)=l$, a given constant vector. This problem is solved in the first part by obtaining the Laplace transforms of a squence of auxiliary problems. If $x_l(p)$ is the Laplace transform of $x(t)$, $F(p)$ the one for $F(t)$ and we set $A=pI-A_0$ (I identity matrix),

$$m_j=A^{-1}A_j\epsilon^{-p\tau_j} \quad (j=1, 2, 3, \cdots, r)$$

$$\sigma_1=m_1 \text{ and } \sigma_j=s_{j-1}s_{j-2}\cdots s_2s_1m_j \quad (j=2, 3, \cdots, r),$$

where $s_j=\sum_{k=0}^{[t/\tau_j]}\sigma_j{}^k$,

then $x_l(p)=(s_r\cdot s_{r-1}\cdots s_2s_1)_lA^{-1}[F(p)+l]$. A^{-1} can be expressed in terms of A^{n-1}, A^{n-2}, $\cdots$, A. In the second part particular solutions are found for $F(t)=Be^{\lambda t}$ (B given constant vector, λ constant); $F(t)=B\cos\omega t+c\sin\omega t$ and $F(t)=\sum_{m=0}^{\infty}(\lambda_m\cos m\omega t+\delta_m\sin m\omega t)$. In the third part, the method is applied to a probem occurring in the analysis of gases. Finally, it is shown how the method can be applied to higher order differential equations.

U. W. Hochstrasser (Lawrence, Kans.)

436:

Campbell, Edwin S. A method for integrating a set of ordinary differential equations subject to a type of numerical indeterminacy. Math. Tables Aids Comput. **11**(1957), 229–233.

The study of free-radical flames involves systems of differential equations $du_j/dZ=F_j(u_l, Z)$, where some of the functions F_j are difficult to evaluate numerically due to a significant loss of accuracy. In an attempt to evade the difficulty, the author sets up a special method of successive approximations, which appears to converge in certain cases. *Walter Gautschi* (Washington, D.C.)

437:

Koval', P. I. On application of the grid method to approximate solution of the differential equation of a vibrating rod. Kiiv. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **19** (1956), 15–31. (Ukrainian)

438:

***Richtmyer, Robert D. Difference methods for initial-value problems.** Interscience tracts in pure and applied mathematics. Tract 4. Interscience Publishers, Inc., New York, 1957. xii+238 pp. $6.50.

For several years, the brilliant collaboration of mathematicians at New York University has been producing some elegant bridges across the gap that has separated the "respectable" abstraction of subjects such as linear operators from the "undignified" empiricism of applied mathematics — represented, in the present instance, by numerical computation. This work has filtered down by now to the text stratum; Bernard Friedman's Principles and Techniques of Applied Mathematics [Wiley, New York, 1956; MR 18, 43] was one instance, and the present work is a second example. That such an effort encounters difficulties in satisfying a wide range of readers is probably clear. In the opinion of the reviewer, the present book is a good deal more successful than most, although there will doubtless be those who find it wanting.

Chapter One of the present book uses examples related to the one-dimensional heat equation to introduce the classical numerical analysis topics of "stability" and "convergence", and some basic results of J. von Neumann and of Francis Hildebrand.

Since many readers will presumably be computing personnel unfamiliar with linear space theory, Chapter Two presents an unusually clear introduction to Banach spaces and their relation to partial differential equation initial-value problems.

The real meat of the book is Chapter Three, a presentation of Peter Lax's beautifully elegant Banach-space approach to the problems of convergence and stability. Lax has successfully obtained Banach-space equivalents that correspond precisely to the classical analysis notions of a "properly posed" problem, a "consistent approximation" by a finite-difference equation, a "convergent" approximation, and a "stable" approximation.

Chapter Four applies the Lax theory to a class of problems where Fourier series or Fourier integral methods can be used. Chapter Five extends the Banach-space treatment to problems involving multi-level difference equations.

The remainder of the book (Chapters VI–X) deals with specific applications, including heat flow and diffusion, the transport equation ("neutron migration and multiplication", "Boltzmann equation"), sound waves, elastic vibrations, and fluid dynamics.

The entire volume makes interesting reading, and one hopes it will convince both pure mathematicians and computer experts that they have a fruitful common area of study. *R. B. Davis* (Syracuse, N.Y.)

439:

Lowan, Arnold N. On the convergence of various iteration processes. Scripta Math. **22** (1956), 222–227 (1957).

One (implicit) difference approximation to the heat equation $u_t=u_{xx}$ is (1) $\Delta_kU(h, k)=r\{\theta\delta_h{}^2U(h, k+1)+(1-\theta)\delta_h{}^2U(h, k)\}$, where $v=\Delta t/(\Delta x)^2$ and θ is a parameter, $0\leq\theta\leq1$. Various iterative solutions of (1) for U at time $(t+1)$ in terms of values at the earlier t are discussed. It is known [see book reviewed above, p. 94] that the scheme (1) is stable for all r when $\frac{1}{2}\leq\theta\leq1$ and for $r\leq1/(2-4\theta)$ when $0\leq\theta\leq\frac{1}{2}$. The naive iterative process for the solution of (1) is shown to be convergent if $r\theta\leq(\frac{1}{4})$, while a more sophisticated one is convergent for all r: proofs depend on a knowledge of the eigenvalues of certain triple diagonal matrices. There are remarks on the comparative rates of convergence of the two processes and on the relation of the processes to the method of relaxation. {The same techniques can be applied to study the full "Gauss-Seidel" process which has better convergence properties.}

John Todd (Pasadena, Calif.)

440:

Groschaftová, Zdenka. Über die Konvergenz der Netzmethode für die Lösung des Dirichletschen Problems und der Wärmeleitungsgleichung. Apl. Mat. **2** (1957), 342–360. (Czech. Russian and German summaries)

Es handelt sich um numerische Behandlungsmethoden zur Lösung der partiellen Differentialgleichungen

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0 \text{ bzw. } \frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial t}$$

mit den Randbedingungen

$$\left.\begin{aligned} u(0, y)&=\varphi_1(y)\\ u(x, 0)&=F_1(x)\\ u(a, y)&=\varphi_2(y)\\ u(x, b)&=F_2(x)\end{aligned}\right\} \text{ bzw. } \left\{\begin{aligned} u(x, 0)&=f(x)\\ f(0)&=f(a)=0\\ u(0, t)&=0\\ u(a, t)&=0\end{aligned}\right.$$

für $0\leq x\leq a$, $0\leq y\leq b$ bzw. $0\leq x\leq a$, $0\leq t\leq\infty$. Die Verfasserin gewinnt hinreichende Bedingungen für die maximale

Schnelligkeit der Konvergenz der sogenannten Netzmethoden .Ist h die Größe der gewählten Netzmasche und sind im Falle des Dirichletschen Problems der Laplaceschen Gleichung die Randfunktionen in allen Ecken des Rechtecks stetig, so gilt: im vorgegebenen Rechteck konvergiert die Netzlösung der Laplaceschen Gleichung gegen die genaue Lösung genau so wie h^2, wenn die ersten Ableitungen der Randfunktionen eine beschränkte Schwankung haben. Diese Konvergenz ist auf dem geschlossenen Rechteck gleichmäßig, wenn die zweiten Ableitungen der Randfunktionen eine beschränkte Schwankung haben. Im Falle der Wärmeleitungsgleichung gesellt sich zum räumlichen Intervall h der Netzmethode das Zeitintervall τ ($\tau/h^2=\beta$). Dann gilt: für alle festen t, $0<t<\infty$ konvergiert die Netzlösung gegen die genaue Lösung genau so wie h^2 unter der Voraussetzung, daß die erste Ableitung der Anfangsfunktion eine beschränkte Schwankung hat. Sie konvergiert wie h^4 unter der Bedingung, daß die dritte Ableitung dieser Funktion eine beschränkte Schwankung hat und $\beta=1/6$ ist. Wenn die zweite Ableitung der Anfangsfunktion eine beschränkte Schwankung hat, konvergiert die Netzlösung gegen die genaue Lösung gleichmäßig für alle t, $0\leq t\leq T<\infty$, wie h^2. Wenn die vierte Ableitung dieser Funktion eine beschränkte Schwankung hat, $\beta=1/6$ ist und die zweite Ableitung der Anfangsfunktion in ihren beiden Endpunkten verschwindet, ist die Konvergenz von der Ordnung h^4.

M. Pinl (Köln)

441:

Val'denberg, Yu. S. The method of solving a certain class of integral equations by casing computers. Avtomat. i Telemeh. **19** (1958), 725–730. (Russian. English summary)

Seidel's iterative method of approximate solution of Fredholm's and Volterra's 1st and 2nd type linear integral equations is worked out. The use of computers to solve special types of these equations is indicated. The results of practical application of this method to automatic control system synthesis problems are considered.

Author's summary

442:

Mikulaschková, Renata. Rounding-off error in numerical calculation from the point of view of statistics. Pokroky Mat. Fys. Astr. **2** (1957), 697–707. (Czech)

The problem of estimation of rounding-off error is studied. The numbers are regarded as observed values of a random variable. The question of rounding-off error in addition is considered first. In this section, an example is presented in which probability theory is used to determine the number of decimal places necessary in the determination of the b_i's in order to have p correct decimal places of $A=\sum_{i=1}^{N} b_i 2^i$ with a given probability q. With the rounding-off error in multiplication two procedures are discussed, one based on the consideration of absolute error, the other on the study of relative error. The reduction of the problem of multiplication to the problem of addition is shown. Examples are given. The effectiveness of statistical approach is demonstrated by comparison with maximalistic theory. Literature dealing with more complicated problems is mentioned at the end of the paper.

J. Janko (Prague)

443:

Smirnov, S. V.; and Potapov, M. K. A nomogram for an incomplete Γ-function and probability function χ^2. Teor. Veroyatnost. i Primenen. **2** (1957), 470–472. (1 insert) (Russian. English summary)

A nomogram of the function $P\equiv P(\chi^2, n)=1-\Gamma(m, y)$ is constructed. It consists of rectilinear and curvilinear scales for n, χ^2, P, $x=(2/n)^{\frac{1}{3}}$, and $t=(2\chi^2)^{\frac{1}{3}}-(2n)^{\frac{1}{3}}$ and can be used for any n and χ^2 equal or larger than one.

S. Kulik (Logan, Utah)

444:

Hovanskiĭ, G. S. Method of construction of nomograms with triangular (hexagonal) transparency. Vyčisl. Mat. **2** (1957), 160–177. (Russian)

In this paper the author gives a method of constructing nomograms for the solution of equations of the type $f(x, y)+g(u, v)+h(s, t)=0$. His nomograms require the use of a specially built transparent triangle instead of a ruler. One adjustment of the triangle to the nomogram is sufficient for taking the readings. *S. Kulik* (Logan, Utah)

445:

James-Levy, J. On the problem of general anamorphosis. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 258–260. (Russian)

A presentation of the equation $z=F(x, y)$ in the form

$$\begin{vmatrix} g_1(x) & f_1(x) & 1 \\ g_2(y) & f_2(y) & 1 \\ g_3(z) & f_3(z) & 1 \end{vmatrix}=0$$

is used in the theory of nomography. The functions $g_1(x)$, $f_1(x)$, $g_2(y)$, $f_2(y)$, $g_3(z)$ and $f_3(z)$ define the scales of a nomogram for solving the equation. The author gives a solution of the problem. *S. Kulik* (Logan, Utah)

446:

***Salzer, Herbert E.; Richards, Charles H.; and Arsham, Isabelle. Table for the solution of cubic equations.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xv+160 pp. (numbered 2–161). $7.50.

This gives the three roots of $\theta y^3+y-1=0$, mostly to 7D, with an accuracy of about one unit in the last place for $\theta^{-1}=-0.001(-0.001)-1$, $\theta=-1(0.001)1$, $\theta^{-1}=1(-.001).001$. For θ negative and near zero 7S are given. First and/or second differences are given where appropriate. The roots were calculated by hand in the range $(-4/27)<\theta<0$ when all are real; outside this range the calculations were performed on an automatic computer (ERA 1103).

There are discussions of the reduction of the general cubic to the above form, of previous tables, and of interpolation, together with illustrative worked examples. The printing of the table is excellent.

John Todd (Pasadena, Calif.)

447:

***Table of natural logarithms for arguments between five and ten to sixteen decimal places.** National Bureau of Standards. Applied Mathematics Series, No. 53. For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C., 1958. xiii+506 pp. $4.00.

Entries of $\mathrm{Log}_e x$ to 16 decimal places are provided for values of x from 5 to 10 at intervals of 0.0001.

This table is a reissue with minor corrections of the fourth volume of "Table for Natural Logarithms" which was prepared by the Mathematical Tables Project of the Federal Works Agency, Work Projects Administration for the City of New York, under the sponsorship of the National Bureau of Standards. It was originally published in 1941 [for review, see MR 3, 275].

D. E. Muller (Urbana, Ill.)

448:

Beattie, Curtis L. Table of first 700 zeros of Bessel functions $-J_1(x)$ and $J'_1(x)$. Bell System Tech. J. **37** (1958), 689–697.

This is a 5-decimal table of the 700 zeros of smallest modulus of the Bessel functions $J_1(x)$ of integer order, and of their first derivatives $J_1'(x)$ arranged collectively in order of increasing magnitude. Zeros of $J_0'(x)=-J_1(x)$ are counted twice, since a code of letters and numbers is attached to each zero — apparently in ignorance of the simpler standard notation — that is, $j_{1,s}$ or $j_{1,s}'$ — for such zeros. *J. C. P. Miller* (Cambridge, England)

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See also 410, 802.

449:

***Malengreau, Julien. Etude des écritures binaires.** Bibliothèque scientifique, 32. Editions du Griffon, Neuchâtel, 1958. 179 pp. (1 plate) 20 Swiss francs.

The author restates several problems in number theory in binary notation, and in particular, points out that the factorization of 2^x-1 is equivalent to finding a pair of numbers p and y, such that the product py is represented by a succession of x ones. Given p, it is possible to calculate the smallest x and y digit by digit, starting with the zero-order digit, and at each step selecting the next digit of y so as to make the corresponding digit in py equal to 1.

The author's "Factorizer" is an electromagnetic computer which, when p can be written in the form

$$p=2^a\pm 2^b\pm 1$$

for $27>a>b$, will carry out the computation automatically. The machine consists of three rings of 27 binary storage elements each. These rings store respectively y, $2^a y$, $2^b y$, presenting the registrations in succession to a set of reading and writing devices. The limitation to three terms in the expression for p results in a simple computing circuit at the read-write point, since in this case the carry from one digit column to the next never exceeds 1.

Much of the book is given over to the derivation of properties of the standard binary notation, and to those of numbers of the form

$$2^a\pm 2^b\pm 1.$$

Tables of special numerical results are appended.

An extremely brief description of the functional operation of the Factorizer is included, but with no technical information beyond the remark that the storage numbers are mechanical with electromagnetic control.

G. R. Stibitz (Cambridge, Vt.)

450:

***Phister, Montgomery, Jr. Logical design of digital computers.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xvi+408 pp. $10.50.

"It is the function of ··· the Logical Design Group, to take the specifications set out by the System Design Group, and the components perfected by the Circuit Design Group, and produce··· diagrams which show how to fit the components together to realize the specification."

Phister has worked on the logical design of several large and small computers, and approaches the subject with an understanding of the practical problems involved in such design. Fortunately, however, his practicality is not of the short-sighted kind that limits itself to rule-of-thumb procedures. Thus, in a full chapter, he lays an adequate foundation for the Boolean algebra that is used extensively in computer circuit design. Here a set of postulates is introduced, and the commonly used formulas of the algebra are derived with satisfactory rigor. Illustrative material and a variety of diagrammatic methods give the reader an understanding of the subject that should be far more secure and fruitful than would be a mere training in manipulation of symbols. Another chapter deals with methods for the simplification of Boolean functions. This topic is of extreme practical importance to the logical designer, since such simplification is translatable into savings in equipment. Two chapters treat the general problem of reducing verbal statements of requirements for the computer into algebraic form, with particular attention to memory devices. The author then turns to the logical or Boolean characteristics of various memory devices, such as magnetic tapes, drums and cores, and of input and output devices. A chapter on the arithmetic unit contains a careful analysis of the requirements for this device, including number-representation, addition rules, complement subtraction, multiplication and division. A further chapter examines self-checking or error-detecting and error-correcting circuits.

The control unit is discussed, with emphasis on the effect that the design of this unit will have on programming. Numerous helpful comments bring out the need to consider the operator's convenience in the design of a control. As the author points out, design methods are tremendous savers of time and effort, but the 'flash of genius' is also important. This is particularly true in the design of the control section. Standardized methods can summarize and make facile application of what is known and recognized in the field of design, but they can never of themselves explore unknown territory, and the designer in conjunction with the system analyst, should always be alert for untravelled paths.

This reviewer considers Phister's book an outstanding contribution to a relatively new subject, and would be hard-pressed to find any vital adverse criticism.

G. R. Stibitz (Cambridge, Vt.)

451:

Mihaĭlov, G. A. The analysis of structure of serial electronic digital computers. Avtomat. i Telemeh. **18** (1957), 1109–1119. (Russian. English summary)

The author's conclusions are as follows.

1. The process of performance of serial computing machines with linear delay line storage units is characterized by a coefficient of use of time η, which is always equal to about 30–40 percent. Therefore, for increasing the speed of execution of commands, those measures are most effective which increase this coefficient by decreasing waiting time.

2. The technical effectiveness of a serial computer is characterized by a coefficient of use of structural elements θ. Using this coefficient, it is possible to judge the effectiveness of various logical designs.

3. The speed of execution of commands and, consequently, the speed of computation depends strongly on the time τ of the large cycle of the memory unit. There is an optimal value of the large cycle time which guarantees the most effective use of the structural elements.

4. In serial fixed-point computers, it is expedient to use a three-cycle, two-address system of commands, because it guarantees the highest speed of computation and the best use of elements with comparison to other systems."

The arguments are supported by numerous graphs, and formulae for inter-relations between the storage and machine parameters are developed.

J. W. Carr, III (Ann Arbor, Mich.)

452:

Hirschhorn, Edwin. Simplification of a class of Boolean functions. J. Assoc. Comput. Mach. **5** (1958), 67–75.

Under a given coding of a digital computer, the fundamental instructions (e.g. "add", "multiply", etc.) correspond to minimal Boolean polynomials. Then the problem of gating involves finding a number of Boolean functions, each of which includes certain minimal polynomials and excludes others. A design objective is that these Boolean functions can be simply represented as logical sums of minimal polynomials.

The degree to which the above design objective can be met depends upon the choice of code for the fundamental instructions. Using heuristic arguments, the author develops two rules-of-thumb for the choice of code, claiming that their application yields favorable results in practice.

G. F. Rose (Santa Monica, Calif.)

453:

***Automatic coding.** Proceedings of the symposium held January 24–25, 1957, at The Franklin Institute in Philadelphia. Journal of The Franklin Institute Monograph No. 3, Philadelphia, Pa., 1957. vii+118 pp. $3.00.

This monograph describes eight approaches to "automatic coding," a method of having a digital computer or Universal Turing machine accept problems written and described in some language other than that of its own hardware.

Eight or more different systems, each designed for a specific machine, are described: "Generalized Programming" — R. M. Peterson; "A–2" and "B–0" — C. Katz; "Print 1" — R. W. Bemer; "B–0" — H. M. Kinzler and P. M. Moskowitz; "Omnicode" — R. C. McGee; "Univac Matrix Compiler" — L. C. McGinn; "IT" — A. J. Perlis and J. W. Smith; and "NCR–304" — E. C. Yowell.

This monograph gives a good U.S.A. sampling of a very transitory area of computer effort as of the Symposium date of which it is a record, January 1957. However, most of the procedures are non-mathematical and thus apparently very liable to obsolescence. The prime exception is the brief discussion of the "IT" (Internal Translator) Compiler of Perlis and Smith, which possesses more permanent value as a nearly-complete translational recursive language structure, the first of its kind. The IT procedure should be part of any background reading involved with any of the areas of automatic theorem proving, language translation, game-playing, etc. — any of the areas which involve recursive attacks on problem solving. Yowell's paper discusses machine hardware, not a coding system.

There is a discussion from the audience on various papers, which is interesting from the user's point of view.

J. W. Carr, III (Ann Arbor, Mich.)

454:

Aufenkamp, D. D.; and Hohn, F. E. Analysis of sequential machines. I. R. E. Trans. **EC-6** (1957), 276–285.

A sequential machine consists of a finite set of machine "states", a finite set of inputs, and a finite set of outputs. The inputs operate on each state to produce a unique output and a unique result state. Two states are termed equivalent if, starting at each state, identical input sequences produce identical output sequences. A "connection matrix" is used to describe the input-output-state change relation, and a definite algorithm of operation on this matrix is produced to find the equivalence classes. A "reduced machine", with its states the equivalence classes of the states of the older machine, produces in a certain sense an "equivalent machine" with a minimum number of states, unique to within a permutation.

Several theorems relative to this reduction process are proved, and a complex example is worked out. It is proposed that this algorithm be programmed for use on a digital computer to reduce labor on the design of such devices.

J. W. Carr, III (Ann Arbor, Mich.)

455:

Scott, A. E. Automatic preparation of flow chart listings. J. Assoc. Comput. Mach. **5** (1958), 57–66.

This paper describes an algorithm (digital computer program) that examines other digital computer programs and translates them into a geometric formulation (flow or sequence diagram). The computer used was the IBM-705, but the process could readily be performed on any other similar general purpose digital computer with sufficiently flexible output printing facilities.

The output is relatively crude, but readable. The algorithm for performing this symbolic-to-pictorial translation is described verbally in this paper, with examples.

The author also discusses extensions of this type of procedure to the automatic printing of diagrams of electronic (in particular, digital computer) circuitry.

J. W. Carr, III (Ann Arbor, Mich.)

456:

Niemz, Werner. Anwendung elektronischer Digitalrechner zur Lösung flugmechanischer Probleme. Z. Flugwiss. **6** (1958), 47–52.

Two digital computer programs for the solution of the differential equations of the dynamics of aircraft, including use of automatic pilots, are described. The first program obtains the characteristic values of the stability matrix from the basic flight parameters; the second determines the responses to various input disturbances. A very simple system of difference equations is used; no error analysis is reported; and a complete set of results for one case is given. The flow diagrams for solution of the problem on the IBM Type 650 computer are included.

J. W. Carr, III (Ann Arbor, Mich.)

457:

Raymond, F.-H. Les calculatrices numériques universelles. Mém. Artill. Franç. **29** (1955), 729–752.

This paper is an introduction to the use of an internally stored programmed digital computer. A description of the characteristics of the main components of a machine of this type, and the techniques used in expressing a mathematical problem in terms of a computer language, form the major part of the discussion. Several examples of machines codes are included to illustrate the formation of loops and the modification of addresses. One section considers the relative merits of one-address and multiple-address codes, of parallel and serial arithmetic units and methods of increasing the speed of operation of computers. The latest techniques in automatic coding and computer components are not included here, as would be expected from the date of this article.

The author is the Directeur de la Société d'Electronique et d'Automatisme, and the discussion is related to the preliminary studies connected with the design of the Calculatrice Universelle Binaire de l'Armement, which is being installed in the Laboratoire Central de l'Armement at the Fort de Montrouge à Arcueil.

M. Lister (College Park, Pa.)

458:

Bashkow, T. R. A "curve plotting" routine for the inverse Laplace transform of rational functions. J. Assoc. Comput. Mach. **5** (1958), 52–56.

This paper is concerned with tabulating the inverse Laplace transforms of a type of rational function arising in problems on circuitry. The work was run on an I.B.M. 650 and the calculations are carried to eight significant figures. The method has proved satisfactory for the type of problem considered and takes ten to fifteen minutes to run on the computor. *R. Wilson* (Swansea)

459:

Stegun, Irene A.; and Abramowitz, Milton. Generation of Bessel functions on high speed computers. Math. Tables Aids Comput. **11** (1957), 255–257.

Recurrence relations, such as those for generating the Bessel functions $J_n(x)$ and $Y_n(x)$ for fixed x and varying n, must be used with care to obtain accuracy and avoid instability. Various procedures are suggested, with particular reference to Bessel functions, for finding starting values and using recurrence relations. Emphasis is on automatic computation, and the method is effectively a mechanisation of a device applied by J. C. P. Miller with desk machines. *L. Fox* (Teddington)

MECHANICS OF PARTICLES AND SYSTEMS

See also 135, 136, 724, 725, 729, 749, 756, 808.

460:

Kaempffer, F. A. On possible realizations of Mach's program. Canad. J. Phys. **36** (1958), 151–159.

Living on a cloud-covered earth and without astronomical observations, some Newton would even then have found the laws of classical mechanics and established the existence of exceptional systems of coordinates or "inertial frames" which, when the clouds disappeared, would be found to be at rotational rest relative to the system of the stars. This is a striking illustration (due, I think, to H. Poincaré) of a fact which, with gravitation as apparently the only interaction between all bodies having inertia, makes it extremely tempting to attempt a realization of Mach's program, viz. to understand all inertial effects as being caused by gravitational interaction.

The author's attempt is a gravitational force

$$\mathbf{F}=m\mathbf{f}+(1/c)(\mathbf{v}\times\mathbf{g}),$$

where the vector fields $\mathbf{f}$ and $\mathbf{g}$ are produced by gravitating masses and currents, with the further assumption that all forms of energy are sources of gravitation. Proceeding by successive approximations and Newton as zero approximation, the first is sufficient to derive Newton's second law for bodies accelerated linearly and to understand Coriolis force when the body is centrally accelerated relative to the universe. Centrigufal forces in rotating bodies require a study of the next approximation.

Concluding, the author discusses the implications of the assumptions of anti-matter having negative gravitational mass [Morrison, Gold]. *E. B. Schieldrop* (Oslo)

461:

Kolčin, N. I. The analytic foundations of the differential method for the investigation of geared linkages. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanizmov **16** (1957), no. 64, 26–53. (Russian)

462:

Artobolevskiĭ, I. I. On certain new mechanisms for drawing plane curves. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanizmov **16** (1957), no. 64, 54–64. (Russian)

463:

Artobolevskiĭ, I. I. On the use of mechanisms for drawing plane curves. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanizmov **16** (1957), no. 64, 65–76. (Russian)

464:

Nadolschi, V. Sur les configurations d'equilibre des systèmes holonomes conservatifs. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 161–164. (Romanian. Russian and French summaries)

465:

García, Godofredo. Variability of mass, total intrinsic energy, differential equations of motion, without recourse to the theory of relativity of A. Einstein, in the case of velocities greater than the velocity of light. Actas Acad. Ci. Lima **20** (1957), no. 4, 1–7. (Spanish)

466:

García, Godofredo. Variability of mass. Total intrinsic energy. The differential equations of motion without recourse to the theory of relativity of A. Einstein. Actas Acad. Ci. Lima **20** (1957), no. 3, 1–22. (Spanish)

467:

***Scarborough, James B. The gyroscope: theory and applications.** Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1958. xii+257 pp. (1 plate) $6.50.

This book contains a good and interesting account of the applications of gyroscopes. The principles of operation of gyrocompasses, stabilizers of ships and monorail cars, and gyroscopic grinding mills are shown, and the essential theory is given, with realistic numerical illustration. Some other applications, e.g. steering of torpedoes, are handled more shortly, and there is a section on the precessional motion of the earth's axis.

However, the reviewer does not recommend the author's dynamics, i.e., the establishment of the equations of motion. He has chosen to expound ab initio such rigid dynamics as he needs, and treats explicitly only the angular momentum principle for motion about a fixed point (§ 15), but later (§§ 43, 72) he handles, without comment, bodies for which no point is fixed. Moreover, his treatment of the limited case is defective in that it makes no appeal to the action-reaction law for the forces between the constituent particles of the body, or to d'Alembert's principle — which as stated in § 11 is only a definition. The gyroscopic reaction moment is stated in § 27 to be (i) a (real) couple exerted by the gyroscope on

its bearings, and (ii) an inertial reaction analogous to (fictitious) centrifugal force, while in § 28 it is spoken of as (iii) a (real) couple acting on the gyroscope. These things may possibly be regarded as questions of taste and clarity rather than logic; but it is not good logic to assume in Chapter X, without comment, that the formula (86.5) proved in a special case is general, and that the attraction between the moon and earth is not as given by Newton's law. Finally there is a notational 'peculiarity' whereby on occasion numbers are equated to vectors. The publishers have done a good job in printing and production.

T. M. Cherry (Melbourne)

468:

Garfinkel, Boris. On the motion of a simple pendulum. Quart. Appl. Math. **16** (1958), 192–196.

Consider the classical problem of a mass attached to the end of a weightless, inextensible and completely floppy string; the other end of the string being attached to a rigid and motionless peg. Allow the system to be under the action of a constant gravitational field and confine motions to the vertical plane. It is elementary to note that if such a circular pendulum is given a horizontal velocity at its lowest position satisfying the inequality $2gl<V^2<5gl$, where l is the length of the string, then the mass will break from a circular to a parabolic path at some level above that of the peg. The author has studied the motions of this system in some generality and has presented relations in terms of dimensionless parameters whereby such motions can be predicted.

H. M. Trent (Washington, D.C.)

469:

Boldinskiĭ, V. I. On a property of the equations of motion of a system with integrals linear in the velocities. Trudy Kirgizsk. Sel'skohozyaĭstv. Inst. **8** (1955), 285–289. (Russian)

The author considers the differential equations of motion of a dynamical system with ignorable coordinates

$$(1)\qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\zeta}=E_\zeta(L),\ \zeta=m+1, m+2, \cdots, n,$$

subject to the m holonomic constraints

$$(2)\qquad f_\alpha(q_1, q_2, \cdots, q_n, t)=0,\ \alpha=1, 2, \cdots, m$$

where $L(q_1, q_2, \cdots, q_n; \dot{q}_{m+1}, \dot{q}_{m+2}, \cdots, \dot{q}_n, t)=T-V$ is the kinetic potential,

$$(3)\qquad E_\zeta(\ )=\partial(\ )/\partial q_\zeta+\sum_{\mu=1}^{m}\partial_{\mu\zeta}(\ )/\partial_\mu$$

and the coefficients $b_{\mu\zeta}$ are taken from the formulas

$$(4)\qquad \dot{q}_\mu=\sum_{\zeta=m+1}^{n+1} b_{\mu\zeta}\dot{q}_\zeta,\ \mu=1, 2, \cdots, m;\ q_{n+1}=t=1,$$

which are obtained by differentiating the constraints (2) with respect to t and solving for the dependent velocities. A coordinate q_k for which the operator $E_k(L)$ vanishes is called cyclic by the author. In analogy with the theorem valid for canonical equations in independent variables (E. T. Whittaker, Analytical Dynamics 1937, § 150, pp. 351–362) the following assertion is proved. The differential equations of motion of a system with ignorable coordinates, in case the coordinate q_k is cyclic, admits an integral which is linear in the velocities. If in general the system admits an integral which is linear in the velocities, then by a change of variables it can be brought into a form containing a cyclic coordinate.

A. K. Nikitin (RŽMat **1956**, # 2177)

470:

Schmieden, C. Nichtlineare Schwingungen bei zwei Freiheitsgraden. II. Ing.-Arch. **26** (1958), 110–128.

This paper is a continuation of the previous one [Ing.-Arch. **25** (1957), 292–302; MR **19**, 591] and is dedicated to the 60th birthday of Heinrich Graf. One considers the problem of nonlinear vibrations of the system with two degrees of freedom, neglecting the damping. To diminish the calculations, the author first treats Duffing's non-homogeneous differential equation ($\ddot{q}+\lambda q+\beta\lambda q^3=F\lambda\sin\Omega t$, $\lambda=\omega_0^2$). Because of the analogy between this and the problem with one and two degrees of freedom, the results are used also in the second case. The ultraharmonic and subharmonic cases are treated. The response curve in the proximity of the resonance of the higher harmonic of the 3rd degree (the 1st ultraharmonic) is discussed (namely, it is shown that the minimum points must lie between $\omega_0/3$ and ω_0). The subharmonic case of the degree 1/3 is treated also.

Further, one treats the basic problem of the system with two degrees of freedom: $\ddot{\vartheta}+\alpha^2\ddot{\varphi}+\lambda\vartheta+\beta\lambda\vartheta^3=F\lambda\sin\Omega t$: $\ddot{\vartheta}+\ddot{\varphi}+\lambda\varphi+\beta\lambda\varphi^3=0$. The A response curve as a function of Ω/ω_0 is plotted. The basic solution for nonlinear case is treated also, especially the case $\alpha=4/5$, i.e., for small vibrations $\Omega/\omega_0=1/3$.

D. Rašković (Belgrade)

471:

Vorob'ev, Yu. V. The method of moments in the problem of vibrations of linear systems. Vyčisl. Mat. **1** (1957), 23–33. (Russian)

472:

Lisovskaya, M. S. Über die Raketenbahnen, die den Mond umfassen. Byull. Inst. Teoret. Astr. **6** (1957), 550–565. (3 inserts) (Russian. German summary)

In dieser Arbeit werden die Existenzbedingungen der Raketenbahnen, die die Erde und den Mond umfassen und neben dem letzteren einen willkürlich kleinen Minimalabstand haben, ausgesagt. Mittels der vom Autor gegebenen grafischen Methode werden 5 solchen symmetrischen Raketenbahnen konstruiert.

Zusammenfassung des Autors

473:

Velasco de Pando, M. Artificial satellites and interplanetary voyages. Rev. Acad. Ci. Madrid **52** (1958), 11–61. (Spanish)

The author studies certain aspects of the theoretical situations arising concerning artificial satellites and interplanetary voyages. The methods used are those of elementary differential equations. Various well-known formulas for the escape velocity of a projectile are derived. The equations of motion of a rocket are obtained. Kepler's laws of planetary motion are discussed in detail, and certain extensions are made. Finally actual computations are given for the most economical means of propulsion of a rocket.

J. De Cicco (Chicago, Ill.)

474:

Roberson, Robert E. Air drag effect on a satellite orbit described by difference equations in the revolution number. Quart. Appl. Math. **16** (1958), 131–136.

When air drag is considered, a point by point numerical calculation of a satellite orbit over thousands of revolutions is impractical. The author solves the equations of motion of the satellite over one cycle, using an approximation that becomes exact as the air drag tends to zero. Taking the final conditions of one cycle as the initial

conditions of the next, he in effect studies the behavior of the elements of the orbit rather than the orbit itself, and gets a more practical computational procedure.

E. Pinney (Berkeley, Calif.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 348.

475:

***Massignon, Daniel. Mécanique statistique des fluides: Fluctuations et propriétés locales.** Travaux et Recherches Mathématiques. I. Publié avec le concours du C. N. R. S. Dunod, Paris, 1957. xiv+263 pp. 3900 francs.

This book presents in a small space a great deal of information which is not readily available elsewhere, and ranks as an original contribution to a rapidly growing subject with important practical applications. The 'statistical mechanics' of the title has its modern literal interpretation, which is not limited to the theory of thermodynamic equilibrium, but includes the molecular theory of irreversible processes. The 'fluids' include gases and liquids, and mixtures thereof, but not specifically plasmas though much of the discussion is applicable to them.

The basic system is a definite number of point particles with definite positions and momenta, interacting with one another in pairs, and also with an external field of force. The statistics of an ensemble of such systems is treated by modern distribution theory, especially the theory of Schwartz to describe, for an individual system, the singular densities in co-ordinate and momentum space. The relations between the microscopic and macroscopic properties of fluids are discussed in detail with the help of this conceptual apparatus.

The book is more concerned with fundamental theoretical questions than practical methods of calculation, but anyone concerned with the properties of fluids is likely to find much that is interesting and useful in it. It is written throughout with gallic elegance, logic, and economy of mathematical thought. Though obviously intended mainly for the graduate and specialist, its difficulty is apparent rather than real, and it deserves to be widely studied. An English translation would surely serve a very useful purpose. *H. S. Green* (Felixstowe)

476:

Rytov, S. M. Correlation theory of thermal fluctuations in an isotropic medium. Soviet Physics. JETP **6** (1958), 130–140.

The theorem of Callen and Welton [Phys. Rev. (2) **83** (1951), 34–40; MR **13**, 477] relating the frequency spectrum of the fluctuations in a thermodynamical variable to the dissipation in the system is applied to an isotropic visco-elastic medium. Space and time correlations are obtained for stress, deformation, velocity, temperature and entropy fluctuations assuming linearized equations throughout. *D. Falkoff* (Waltham, Mass.)

477:

Biedenharn, L. C. A note on statistical tensors in quantum mechanics. Ann. Physics **4** (1958), 104–113.

A parametrization of the quantum mechanical density matrix for mixed angular momentum states is given. The conditions on these parameters for the density matrix to represent a pure state are derived.

D. Falkoff (Waltham, Mass.)

478:

Cowan, Robert D.; and Kirkwood, John G. Quantum statistical theory of plasmas and liquid metals. J. Chem. Phys. **29** (1958), 264–271.

A Debye-Hudsel statistical theory of Coulomb interactions in highly-ionized plasmas (temperatures of order $kT \sim 10$'s of electron volts) is presented and calculated numerically. The electrons are treated by a temperature-dependent Thomas-Fermi theory. It is not clear that a great improvement over previous theories has been achieved. *P. W. Anderson* (Murray Hill, N.J.)

479:

Siegert, A. J. F.; and Teramoto, Ei. Simplified derivation of the binary collision expansion and its connection with the scattering operator expansion. Phys. Rev. (2) **110** (1958), 1232–1234.

Lee and Yang's binary collision method leads to an expansion for the density matrix. The same expansion is obtained here by using Green function equations. It is shown that the binary collision expansion is the Laplace inverse of the coordinate representation of the expansion in terms of scattering operators. *D. ter Haar* (Oxford)

480:

***Münster, Arnold. Statistische Thermodynamik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. xi +852 pp. DM 138.00.

Of the many books on Statistical Thermodynamics which have appeared since the publication of the work of Fowler [Cambridge U. Press 1936], Fowler and Guggenheim [Cambridge U. Press 1939; MR **1**, 192, and Tolman [Clarendon Press, Oxford, 1938], few have attempted (and almost none succeeded) to emulate these treatises in their scholarly, definitive treatments of the theory and its application. Münster's volume admirably fills this need; his book on "Statistische Thermodynamik" is no doubt the most comprehensive (850 pages), incisive, and modern work available on statistical mechanics of equilibrium systems. The chapters on fundamental theory present thorough discussion of classical and quantal systems of independent particles, classical and quantal systems of dependent particles (Gibbsian ensemble theory), and fluctuation theory based on the grand-canonical ensemble. Münster gives an excellent presentation of the problem of ensemble theory versus ergodic theory as a foundation of statistical mechanics and chooses the former as hypothesis to be confirmed by agreement with experimental results. The section concludes with an excellent discussion of distribution functions and various formal relations obeyed by them, a topic which has been prominent in the literature of chemical physics in the last twenty-five years.

The next three sections are primarily devoted to the application of the theory to gases, solids, liquids and solutions, although supplements of theory are freely interspersed. The ideal gas, internal degrees of freedom, chemical equilibrium, the real gas and condensation, the ideal crystal, the third law of thermodynamics, cooperative phenomena and the Ising lattice, pure liquids, solutions of electrolytes, non-electrolytes and polymers are some of the topics treated authoritatively and succinctly.

The book is clearly indispensable to scientists whose work is in the ever increasing but common area between physics and chemistry. Minor criticisms can always be

made but are utterly overshadowed by the intrinsic value of the book. The author closes his work with the quotation from Goethe, "So eine Arbeit wird eigentlich nie fertig, man muß sie für fertig erklären, wenn man nach Zeit und Umständen das Möglichste getan hat." The "utmost effort" is well rewarded by a work which will appear "complete" for many years.

J. Ross (Providence, R.I.)

481:

Hemmer, P. Chr.; and Wergeland, H. Brownian motion of a particle in a linear chain. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 137–143 (1958).

A particle in an infinite chain of oscillators is assumed initially to have a given velocity, all other coordinates and momenta of the system being distributed in a Gaussian manner. It is found that at later times the velocity distribution of the chosen particle is also Gaussian, with an oscillating mean and deviation from the mean. These oscillations ultimately damp out, and the velocity distribution approaches its equilibrium form. These is a formal analogy with the solution of a Fokker-Planck equation, but the authors point out that the relaxation process treated here is not Markovian. The finite chain and the quantum mechanical case are also discussed.

S. Prager (Brussels)

482:

Prigogine, I.; and Henin, F. On the transport equation for dilute gases. Physica **24** (1958), 214–230.

The approach to equilibrium of the velocity distribution $\rho_0(v, t)$ in a dilute homogeneous gas is studied by the method of Brout and Prigogine [Physica **22** (1956), 621–636], and a series development of ρ_0 in powers of ct, the concentration-time product, is obtained. The series is equivalent to a differential equation relating $\partial\rho_0/\partial t$ to a collision operator (actually an infinite series of operators) acting on ρ_0. Using a relation previously established for a two body system by Prigogine and Henin [Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 814–827; MR **19**, 1208], this result is shown to reduce to the classical Boltzmann equation.

S. Prager (Brussels)

ELASTICITY, PLASTICITY

See also 406, 476, 605, 606, 763.

483:

***Freudenthal, Alfred M.; and Geiringer, Hilda. The mathematical theories of the inelastic continuum.** Handbuch der Physik, herausgegeben von S. Flügge. Bd. 6. Elastizität und Plastizität, pp. 229–433. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

Freudenthal's part of this article is mostly concerned with the constitutive equations of the main types of inelastic solids: viscoelastic, plastic, and viscoplastic. The treatment is uneven. The discussion of variational and extremum principles is superficial and makes no reference to work of the last few years; moreover, certain results given for the rigid-plastic solid are erroneous, being based on a serious misconception about the proportionality factor in the flow law (repeated on pp. 251, 254 and 284). By contrast, the section on viscoelasticity (operators, relaxation spectra, dynamical problems, etc.) is authoritative and well balanced. The section on plasticity, however, does not reflect present trends. Thermodynamical principles are well discussed at the outset of the article but, as usual, contribute little to the theory of irreversible straining.

Geiringer's part is limited to non-hardening plastic solids with Mises yield function and potential. The flavour is classical rather than modern. The treatment is generally excellent, though it is occasionally academic (as in the theory of integration of the plane problem) and perfunctory in the account of applications (e.g., torsion, expansion of cavities in shells and plates). There is no mention of recent work on uniqueness in rigid-plastic problems, dating from 1956. The article will be highly valued primarily for the author's personal contributions in the extensive systematic account of characteristics, discontinuities, and singularities. The bibliography is generally very full and scrupulously accurate, and it is therefore surprising to find no reference to the work of D. H. Parsons on the axially symmetric field equations nor to the many papers of A. P. Green on plane strain problems.

R. Hill (Nottingham)

484:

***Goodier, J. N. The mathematical theory of elasticity.** Elasticity and Plasticity. Surveys in Applied Mathematics, Vol. 1, pp. 1–47. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. ix+152 pp. $6.25.

The author states of this work that "Its principal aim is to draw attention to those significant recent developments (in elasticity theory) believed to be least known to readers whose first language is English". This is not to be taken too literally, since many of the works cited are written in English, and Italian works, for example, are notable by their absence. It is primarily a descriptive account of Russian researches in linear elasticity theory and related researches by Western workers. It does not pretend to cover Russian work comprehensively or to give a well proportioned sampling of such researches. The author does manage to convey a reasonable impression of the quality of better Russian researches and to give some idea of the differences between Russian and Anglo-American researches in elasticity. Works covered are generally described in just enough detail to indicate the problem considered and, in some cases, the method of attack, which packs considerable information in short space. The breadth of coverage is sufficient to make it quite useful as a reference work for researchers in elasticity theory, particularly in view of the dearth of surveys in English of Russian work.

J. L. Ericksen (Baltimore, Md.)

485:

***Hodge, Philip G., Jr. The mathematical theory of plasticity.** Elasticity and Plasticity. Surveys in Applied Mathematics, Vol. 1, pp. 49–144. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. ix+152 pp. $6.25.

This is a discussion of selected recent developments in plasticity theory, stressing ideas involved in the formulation of quite general theories and general theorems. The work is divided into seven very short chapters whose titles provide a rather clear picture of the contents. These are Theory of perfectly plastic solids, Theory of strain-hardening plastic solids, Piecewise linear plasticity, Bending of a circular plate, Other problems and Russian contributions. The last consists of a highly condensed but readable survey of Russian work. The material is inter-

esting, well organized and neatly presented. Some of the fundamentals are not made entirely clear. For example, on p. 57 we find that "A basic assumption of plasticity theory is that the total strain rate can always be decomposed into an elastic part and a plastic part." No assumption is involved in dividing the strain rate into two parts unless one requires the parts to have some special properties. No such properties are mentioned and the reviewer found no place where they were used. The author's equation (1.8) merely defines the plastic part in terms of total strain rate and stress. Despite this, the work provides a quite readable and unified account of plasticity theory as seen by one of the leading experts. To some extent, selection of material is based on personal interest. It is difficult to see objective reasons for omitting some works, such as that of Green (J. Rational. Mech. Anal. 5 (1956), 725–734; MR 18, 436).

J. L. Ericksen (Baltimore, Md.)

486:

***Кац, А. М. [Kac, A. M.] Теория упругости. [Theory of elasticity.]** Edited by V. K. Prokopov. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 207 pp. 4.65 rubles.

This book in a mimeographed form served for many years as a text at the Lenigrad Polytechnic Institute. The present edition was corrected and somewhat enlarged by V. Prokopov. The book, combining conciseness of presentation and practically complete contents, may serve not only as a text but also as a useful reference for a designer. The general theory and the standard methods are given in Chapters I to V. They are as follows. I. Stresses. II. Deformations. III. Stress-deformation relationship. IV. General equations of the theory of elasticity. V. Statement of the elasticity problems and methods of solution. Chapters VI to XII contain examples of special cases which illustrate the general theory and are of practical interest to a designer. They are as follows. VI. Simple axi-symmetric problems. VII. Bending of thin plates. VIII. Bending of symmetrically loaded cylindrical shells. IX. Torsion of beams. X. Plane problem of elasticity. XI. Problem of bodies in contact. XII. Thermal stresses. Exercises and problems with solutions are not included in the book; the common practice in the USSR is to publish them under separate covers.

In some respects, the theoretical part (Chapters I–V) of this book could be compared with the book by H. M. Westergaard, "Theory of elasticity and plasticity" [Wiley, New York, 1952; MR 14, 513]. In the reviewer's opinion it is inferior to Westergaard's.

T. Leser (Aberdeen Md.)

487:

Ericksen, J. L. Hypo-elastic potentials. Quart. J. Mech. Appl. Math. 11 (1958), 67–72.

Using cartesian tensor notations, the author studies the constitutive equations

$$\text{(A)} \qquad \dot{t}_{ij}-t_{ik}\omega_{jk}-t_{jk}\omega_{ik}=B_{ijkm}d_{km},$$

where t_{ij} is the stress tensor, v_k the velocity vector, a 'dot' denotes material derivative and $2d_{ij}=v_{i,j}+v_{j,i}$, $2\omega_{ij}=v_{i,j}-v_{j,i}$. Also B_{ijkm} is a continuously differentiable tensor invariant of t_{ij} such that $B_{ijkm}=B_{jikm}=B_{ijmk}$. Necessary and sufficient contions are obtained for the existence of scalar invariants ϕ, ψ of t_{ij} such that

$$t_{ij}d_{ij}=\phi\dot{\psi}$$

holds whenever t_{ij} and $v_{i,j}$ satisfy (A).

A. E. Green (Newcastle-on-Tyne)

488:

***Verma, P. D. S. Hypo-elastic strain in a rotating shaft and a spherical shell.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 99–110. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The author considers the problem of finding the stresses in an incompressible hypo-elastic material of grade zero [cf. C. Truesdell, J. Rational Mech. Anal. 4 (1955), 83–133; MR 16, 880] in the special cases mentioned in the title. It is not clear to the reviewer how the results of this paper are to be interpreted because the author makes no clear distinction between displacements and velocities.

W. Noll (Pittsburgh, Pa.)

489:

Tschoegl, N. W. The general relationships between the elastic constants of isotropic materials in n dimensions. Austral. J. Phys. 11 (1958), 154–157.

The author derives relationships between the elastic constants of homogeneous isotropic materials in n dimensions. These formulae reduce to the well-known expressions for $n=2$ and $n=3$.

L. E. Payne (Newcastle-on-Tyne)

490:

Boley, Bruno A. A method for the construction of fundamental solutions in elasticity theory. J. Math. Phys. 36 (1957), 261–268.

Assuming that a linearly elastic body, free from body forces and surface tractions, is subject to a concentrated unit force in the x_i-direction at P_1, denote the resulting displacement component in the x_j-direction at P_2 by $\delta_{ij}(P_1, P_2)$. The solution of any stress boundary value problem can be expressed by integrals involving these $\delta_{ij}(P_1, P_2)$.

The "Green's tensor function" δ_{ij} is the solution of a tensor integral equation of the Fredholm type. The author discusses methods of finding approximate solutions by iteration. He illustrates the procedure with the example of a semi-infinite solid, in which case a comparison with the exact solution is possible.

W. Noll (Pittsburgh, Pa.)

491:

Manfredi, Bianca. Sopra la più generale equazione reologica di stato per una classe di solidi naturali. Boll. Un. Mat. Ital. (3) 12 (1957), 422–435.

The author expresses in the homographic notations used by the Italian school some properties of various tensions of stress and strain. She lists a number of hypotheses which lead to a special and partially linear form for the constitutive equations of a material responding to strain from a possibly varying reference configuration of Eckart's type. {For analysis and criticism of theories of this type, see § 82 of the reviewer's paper [J. Rational Mech. Anal. 1 (1952), 125–300; 2 (1953), 593–616; MR 13, 794; 15, 178].}

C. Truesdell (Bloomington, Ind.)

492:

Adkins, J. E. A reciprocal property of the finite plane strain equations. J. Mech. Phys. Solids 6 (1958), 267–275.

The author considers the general equations of non-linear elasticity for plane strain of incompressible isotropic and transversely isotropic materials. From any given solution, he shows that another solution can be obtained by interchanging material and spatial coordinates and, in cases where there is homogeneous strain perpendicular to this plane, by introducing a suitable scale factor. A

slightly more general result is obtained for Mooney materials. Illustrative examples are included.

J. L. Ericksen (Baltimore, Md.)

493:

Fuka, Jaroslav. Lösung der ersten Aufgabe der Elastizitätstheorie auf dem exzentrischen Ringe. Apl. Mat. 3 (1958), 45–66. (Czech. Russian and German summaries)

The first boundary value problem of the excentric elastic ring is solved by representation of the excentric ring on the concentric ring by which a new boundary value problem is defined, which is then solved by the method of stress-functions expanded into Laurent series, developed by Mushelishvili. The connecting relations are established between the solutions and boundary conditions of the two problems, on the basis of which the existence and uniqueness of the solution of the excentric ring is demonstrated.

A. M. Freudenthal (New York, N.Y.)

494:

Mitra, M. Disturbance produced in an elastic half-space by an impulsive twisting moment applied to an attached rigid circular disc. Z. Angew. Math. Mech. 38 (1958), 40–43. (German, French and Russian summaries)

This paper concerns the displacements in an elastic half space due to an impulsive twisting moment applied to a rigid plate attached to its face. The stresses transmitted from plate to elastic medium are assumed to be those given by Bycroft [Philos. Trans. Roy. Soc. Ser. A, 248 (1956), 327–368; MR 19, 195]. The author derives formulas for the circumferential displacement when the medium is either isotropic or transversely isotropic.

W. E. Boyce (Troy, N.Y.)

495:

Pell, William H. Elastic problem for a ring of uniform force in an infinite body. J. Res. Nat. Bur. Standards 60 (1958), 365–373.

Kelvin's integral representation for displacements in an elastic solid under body force distribution has been used to obtain the displacements for a concentrated force applied along, and normal to a circle in an infinite elastic solid. Spherical coordinates have been employed and results have been expressed in elliptic integrals.

S. C. Das (Chandernagore)

496:

Segedin, C. M. The relation between load and penetration for a spherical punch. Mathematika 4 (1957), 156–161.

The author determines the stress distribution in an elastic half-space caused by the indentation of the plane surface with a rigid spherical punch. This problem was solved previously by the reviewer [J. Soc. Indust. Appl. Math. 1 (1953), 53–71; MR 15, 267] using curvilinear coordinates. The author avoids the introduction of curvilinear coordinates and finds that both the maximum penetration and the load necessary to produce this penetration can be expressed in closed form in terms of the radius of the circle of contact.

L. E. Payne (Newcastle-on-Tyne)

497:

Knops, R. J. On the variation of Poisson's ratio in the solution of elastic problems. Quart. J. Mech. Appl. Math. 11 (1958), 326–350.

If for a three-dimensional boundary value problem of an elastic medium not subject to body forces Maxwell's biharmonic stress functions are known for a particular, not necessarily real, value of Poisson's ratio, the solution for a different ratio can be found by establishing and solving the equations for the differences between the initial and the final stress functions. This method, which has previously been used by Westergaard, is applied to the particular case of an initial value of Poisson's ratio $\nu=\infty$, for which the displacements, stresses and stress-functions are harmonic, and illustrated by the problems of an internal force in a semi-infinite medium with rigidly constrained boundary and a force acting on the common vertex of a body bounded by two right circular coaxial cones.

A. M. Freudenthal (New York, N.Y.)

498:

Mossakowska, Zofia. Concentrated force in the interior of a transversely isotropic elastic semi-infinite space. Arch. Mech. Stos. 10 (1958), 233–251. (Polish and Russian summaries)

The author studies the problem of determining the stresses in a transversely isotropic semi-space under a point concentrated load. Three types of boundary conditions are considered: (1) zero stress; (2) mixed conditions; (3) zero displacements. The author proceeds in the following manner. He introduces the six second order partial differential equations satisfied by the components of the displacement vector. It is stated that these equations can be replaced by single sixth order partial differential equations for the three components of a new "displacement" vector. The physical displacements and stresses are then determined in terms of the new "displacement" vector. By choosing two components of this vector to be zero, the author reduces the problem to the solution of a single fourth order partial differential equation which is solved by use of a double Fourier integral.

N. Coburn (Ann Arbor, Mich.)

499:

***Weber, Constantin; und Günther, Wilhelm. Torsionstheorie.** Friedr. Vieweg & Sohn, Braunschweig, 1958. iv+307 pp. DM 38.00.

The classical problem of torsion of prisms is solved for a number of sections by employing well-known methods. Most of the sections dealt with are rectilinear or circular with holes and slits. The methods include the semi-inverse method, Prandtl's membrane analogy, image systems and conformal mapping on a unit circle. The determination of upper and lower bounds for the shearing stresses and the torsional rigidity are also discussed. There are very few references to modern literature on this subject. The excellent collection of torsion results by T. J. Higgins, papers by Pólya, Szegö, Payne, Synge, Prager on the limits of the torsional rigidity, those by Seth on rectilinear sections, image-method applications by Hay, and a host of other similar papers are not even mentioned.

It is also unfortunate that the applications begin with the semi-inverse method and membrane analogy. This vitiates presentation of the subject as a boundary value problem. For example, though the elliptic section with a confocal hole is treated with the help of a conformal mapping, the ordinary elliptic section is done by the semi-inverse method of assuming a complex function and then finding out what boundary conditions it can satisfy. In like manner, in the chapter on rectilinear sections it is not pointed out that the solution for the equilateral triangular section follows immediately from the conformal mapping technique of Trefftz.

In Chapter 1, the torsion displacements u, v, are derived

geometrically. The boundary condition is obtained and its modifications for multiply-connected regions are discussed in detail. Chapters 2, 3 and 4 employ the membrane analogy and the semi-inverse method to get classical results. Chapter 5 and 6 deal with circular and circular-ring sectors. A number of closed solutions are obtained. The mapping on a unit circle is used in Chapters 7 and 8 to discuss cases of an epicycloid, inverse of an ellipse, a leminscate, eccentric circles and confocal ellipses. Chapter 9 contains interesting results for circular sections with a number of circular slits. Rectilinear sections, both finite and infinite are discussed in Chapters 10 to 13. A number of useful results are obtained for T- and L-sections. Trefftz' method is only applied to finite sections and not to infinite ones. The image method is given in detail in Chapters 14 to 16 with applications to circular and eccentric sections, sections bounded by two curves, triangular, L- and T-sections. Chapter 17 contains solutions for circular sections with one or more slits. Its results can be very useful in technological applications. Small holes and reentrant angles are treated in Chapters 18 and 19. Minimal theorems and bounds for the shearing stress and torsional rigidity are given in Chapters 20, 22, 23. Chapter 21 contains a number of solutions for strip sections with rounded curves. The book ends with two useful appendices on the theory of elasticity, two dimensional potential functions and conformal mapping.

The book is a good collection of a number of torsion solutions useful in engineering problems. Its presentation could have been made more scientific and rigorous. There are a number of misprints, e.g., on page 133 in the bottom line the expression in the brackets should be $(1+z_0^5)$ and not $(1+z_0)$. *B. R. Seth* (Kharagpur)

500:

Stüssi, Fritz. Zur Prandtlschen Membrananalogie der Torsion. Z. Angew. Math. Phys. **9b** (1958), 661–667.

Prandtl's membrane analogy combined with the numerical methods of structural statics is used to solve the torsion problem of a rectilinear section. The torsional rigidity is calculated in two particular cases. The values so obtained agree very well with known results.

B. R. Seth (Kharagpur)

501:

Huan, Kè-Čži. On the fundamental equations of the theory of thin-walled bars with open profile at the elastic limit. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 11, 122–136. (Russian. Chinese summary)

Using a method of comparing the order of quantities, the engineering theory of thin-walled open bars ($2h$= thickness, R=radius of curvature of the cross-section, L=length of the bar) is discussed on the basis of the theory of shells. In the elastic range, the theory holds if (a) $\rho\sim\lambda^{3/4}$, $\lambda\ll 1$ ($\rho=R/L$, $\lambda=h/R$), or if (b) $\rho^2\sim\lambda\ll 1$, and some additional conditions are satisfied by the external load. If in case (b) these conditions are not satisfied, the engineering theory holds in some particular instances only, such as that discussed by Goldenveizer. In case (a), the engineering theory is true also in the elastic-plastic range, and in case (b) if other additional conditions are satisfied by the external load.

J. Nowinski (Madison, Wis.)

502:

Vlasov, V. V. The method of initial functions in problems of the equilibrium of thick many-layered slabs. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 7, 40–48. (Russian)

503:

Chatterjee, B. B. Stresses in an aeolotropic elliptical disc rotating about an axis of symmetry lying in its middle plane. J. Tech. Bengal Engrg. Coll. **2** (1957), 137–139.

The paper gives a brief account of stress calculations (explicit solution) using four elastic constants.

J. Heyman (Cambridge, England)

504:

Kacner, Artur. A closed solution in the case of a semi-infinite plate with discontinuous boundary conditions. II. Arch. Mech. Stos. **10** (1958), 57–80. (Polish and Russian summaries)

As in Part I [ibid **9** (1957), 371–380; MR **19**, 1107] the author considers a uniformly loaded semi-infinite strip with a moment $M(x)$ applied to a specified segment of the short edge, which is otherwise free. The problem is to determine $M(x)$ so that $M(x)$ will be proportional to the deflection along this segment. An infinite system of equations for the Fourier coefficients of $M(x)$ is obtained and solved exactly. *R. C. T. Smith* (Armidale)

505:

Vlasov, B. F. On the equations of the theory of the bending of plates. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 12, 57–60. (Russian)

A derivation of E. Reissner's equations for the bending of a thin plate [J. Math. Phys. Mass. Inst. Tech. **23** (1944), 184–191; MR **6**, 195]. *R. C. T. Smith* (Armidale)

506:

Sokołowski, Marek. The bending of transversally non-homogeneous plates of moderate thickness. Arch. Mech. Stos. **10** (1958), 315–328. (Polish and Russian summaries)

This is a generalization of the Michell-Love development of the theory of moderately thick, isotropic, homogeneous plates, to isotropic, non-homogeneous plates. The latter are characterized by a constant Poisson's ratio and a symmetrical variability of Young's modulus $E(z)$, z denoting the distance from the middle plane of the plate. The boundary conditions at the faces are satisfied at any point, and in an integral manner at the periphery of the plate. Two cases concerning (a) the load acting at the periphery and (b) a uniformly distributed lateral pressure are considered. Stresses and displacements are found. It is shown that in both cases the deflection of the middle plane of the plate satisfies the Lagrange-Germain equation for thin plates. *J. Nowinski* (Madison, Wis.)

507:

Chattarji, P. P. Elastic distortion of a cylindrical hole by tangential tractions varying with depth on the inner boundary. J. Tech. Bengal Engrg. Coll. **2** (1957), 141–144.

The title problem has been discussed for a large thick plate (i) with one face fixed and the other free and (ii) with both faces free when the tractions vary linearly and quadratically respectively. *S. C. Das* (Chandernagore)

508:

Vorovič, I. I. On the existence of solutions in the non-linear theory of shells. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 203–206. (Russian)

The author considers a boundary value problem containing the system of non-linear differential equations controlling large deflections of shells. Besides boundary conditions, there are also a number of additional con-

ditions concerning regions, boundaries and terms and coefficients in the equations. Using functional analysis, the author proves the existence of a unique solution for the problem and deduces properties of this solution.
T. Leser (Aberdeen, Md.)

509:
Iwiński, Tadeusz. Application of the method of separation into factors to the differential equation of a certain plate. Arch. Mech. Stos. **10** (1958), 35–56. (Polish and Russian summaries)

The method described below is used to solve an equation derived by H. M. Berger's method [J. App. Mech. **22** (1955), 465–472; MR **17**, 429] for the problem of finite elastic deflections of a circular orthotropic plate. It is desired to transform

$$\sum_0^{n+1} a_k(x)y^{(k)}=f(x), \quad a_{n+1}\equiv 1,$$

into the form

$$\left[\frac{d}{dx}+a(x)\right]\sum_0^n b_k(x)y^{(k)}=f(x), \quad b_n\equiv 1.$$

The solution of the first equation will then depend upon a quadrature and the solution of an nth order equation. Once $a(x)$ is known, the functions $b_k(x)$ may be found by quadrature. The function $a(x)$ is a solution of an nth order nonlinear equation, called one of Riccati type by the author since, for $n=1$, it is one. The cases where this Riccati type equation degenerates to an algebraic one are considered, for example, when the original equation has constant coefficients or is an Euler equation.

{The claims made by the author for his method are, perhaps, too sweeping, e.g., "in the light of this theory, ... the method of variation of constants becomes superfluous."}
N. D. Kazarinoff (Ann Arbor, Mich.)

510:
Karas, K. Die Auswölbungen der Kreisringmembran unter hydrostatischem Druck. II. Inhomogener Spannungszustand. Ing.-Arch. **26** (1958), 157–180.

Continuation of paper reported in MR **9**, 594.

511:
Grigolyuk, È. I. Finite deflections of three-layered shells with hard core. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 1, 26–34. (Russian)

512:
Mathews, P. M. Vibrations of a beam on elastic foundation. Z. Angew. Math. Mech. **38** (1958), 105–115. (German, French and Russian summaries)

This paper deals with the vibrations of an infinite beam on an elastic foundation, subjected to an alternating load whose point of application moves along the beam with velocity v. Solutions of the form

$$y(r, t)=y_1(r)\cos \omega t+y_2(r)\sin \omega t,$$

where $r=x-vt$, are sought; the coefficients y_1 and y_2 are expressed as the inverse Fourier transforms of ratios of fourth degree polynomials in the transform variable. The author has carried out the inversion for the general case of an undamped motion, as well as for several special cases. Of particular interest is the determination of combinations of velocity v and frequency ω which, in the absence of damping, lead to unstable motion of the beam.
W. E. Boyce (Troy, N.Y.)

513:
Jullien, Yves. Vibrations d'une plaque rectangulaire simplement soutenue au pourtour et soumise à une charge ponctuelle. C. R. Acad. Sci. Paris **246** (1958), 1371–1374.

Vibrations of a rectangular plate resting on an elastic foundation, hinged at the edges and supporting a point load in addition to a uniform load are investigated by use of Fourier series. A complicated equation for the natural frequencies is obtained and numerical results for a few particular cases are given.
R. C. T. Smith (Armidale)

514:
Aben, H. K. The elastic stability and postbuckling behaviour of a long cylindrical panel under shear. Izv. Akad. Nauk. Eston. SSR. Ser. Tehn. Fiz.-Mat. Nauk **1958**, 3–6. (Russian. Estonian and English summaries)

Approximate methods are used to discuss the problem stated in the title. Some graphs and numerical results are given.
R. C. T. Smith (Armidale)

515
Yusuff, Syed. Buckling phenomena of stiffened panels. J. Aero. Sci. **25** (1958), 507–514.

A study is made of the instabilities that occur in stiffened panels forming compression surfaces. Only those buckling modes of the panels that are due to the flexural deformation and rotation of the stiffener without distortion in its own plane of the cross section are considered in the analysis, as they are of primary interest in practical stiffened panel design for aircraft. References are given to studies of other types of instabilities.

The modes of instabilities associated with the deformations of the stiffeners involve plate buckling phenomena and are characterized by buckles of different wavelengths. If the major deformation is flexural, the panel will have buckles of long wavelength. If the stiffeners twist without significant bending, the buckles will be of short wavelength. In either of these cases, failure will be coincident with the onset of buckling. However, a panel may have mixed instability when one buckling mode of short wavelength due to negligibly small rotations of the stiffeners is followed by a buckling mode of long wavelength and failure. Based on these considerations, equations are derived for the critical stress. Charts are given for use in computing these stresses.

Extensive comparisons are made between results computed by the theory and those obtained from tests of z-section stiffened panels and panels of integral construction with unflanged stiffeners covering a wide range of configurations. Good agreement is obtained.
W. D. Kroll (Washington, D.C.)

516:
Babič, V. M.; and Alekseev, A. S. On the ray method of calculation of the intensity of wave fronts. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958**, 17–31. (Russian)

The authors consider solutions $U(x, y, z, t)$ of (i) the scalar wave equation with variable velocity c, or (ii) the vector differential equations of elasticity, having discontinuities on a wavefront $t=\tau(x, y, z)$. Such solutions they write in the form

$$(*)\quad U(x, y, z, t)=U_0(x, y, z)f_0(t-\tau)+U_1(x, y, z)f_1(t-\tau)+O(f_2(t-\tau)),$$

where $f_2'(t)=f_1(t)$, $f_1'(t)=f_0(t)$ and $f_2(t)$ is in some sense negligible relative to its derivative. Under the principal assumption that the coefficients of $f_0''(t)$ and $f_0'(t)$ vanish when (*) is substituted in (i) or (ii), they deduce first-

order differential equations for determining U_0. On integration they obtain expressions for the intensity, a typical one of which is $U_0=(c/J)^{\frac{1}{2}}f(\alpha_1, \alpha_2)$, where α_1, α_2 are orthogonal coordinates in a wave surface and J is a function of the derivatives along these coordinates. The function f is an arbitrary function, somewhat like a source function, the determination of which for particular cases constitutes the greater part of the paper. The reflection of waves from curvilinear boundaries, propagation of waves in media with one-dimensional inhomogeneity, and diffraction of cylindrical waves by a wedge are taken up in detail to illustrate the method.

R. N. Goss (San Diego, Calif.)

517:

Chopra, S. D. On the equivalence of saddle point approximations and ray theory in elastic wave problems. Geophys. J. 1 (1958), 164–179.

Let M be an internal stratum, of plane parallel boundaries, lying in welded contact between halfspaces M_1 and M_2; all three media are perfectly elastic, homogeneous and isotropic; M_1 and M_2 are of identical elastic properties. Let O lie in M, halfway between M_1 and M_2. Then the problem of the elastic waves produced by a harmonic point source of dilatational waves at O is examined, using van der Pol and Bremmer's work [Phil. Mag. 24 (1937), 825–863] on the method of steepest descents and saddle-point approximations. Detailed comparison is made with the ray theory approach. Plane P and SV seismic waves incident against a plane boundary are considered, and reflection and refraction coefficients set down.

K. E. Bullen (Sidney)

518:

Roseau, Maurice. Sur une équation intégrale de la théorie de la diffraction des ondes élastiques. C. R. Acad. Sci. Paris 245 (1957), 2013–2014.

The author considers the integral equation of Wiener-Hopf type

$$i\tau\cos\varphi\exp(i\tau x\cos\varphi)=\int_0^\infty\{\partial^2(G_2-G_1)/\partial x^2+\sigma^2G_2\}f(x')dx'=0 \quad (x>0),$$

where $G_1=\frac{1}{4}iH_0^1(\tau|x-x'|)$, $G_2=\frac{1}{4}iH_0^1(\sigma|x-x'|)$ and σ and τ have positive imaginary parts. The solution is carried through by the usual technique, and it is concluded that $f(x)$ behaves like $x^{-\frac{1}{2}}$ for small x, and is exponentially small for large x.

F. Smithies (Cambridge, England)

519:

Berry, D. S. Stress propagation in visco-elastic bodies. J. Mech. Phys. Solids 6 (1958), 177–185.

The author uses the fact that the Laplace transforms of the governing equations for the stress and displacement in a viscoelastic solid are identical to those for an elastic solid, except that the elastic constants are replaced by the viscoelastic moduli, to solve the problems of forced torsional oscillations in a circular cylinder and of a time dependent uniform pressure acting on the surface of a spherical cavity in an infinite medium. The former solution gives a simple method of determining the deviatoric complex modulus.

D. R. Bland (Manchester)

520:

Sentis, André. Sur le mouvement stationnaire dans les phénomènes interfaciaux. C. R. Acad. Sci. Paris 245 (1957), 1776–1778.

The equations of motion of a simplified visco-elastic medium without viscosity or elasticity developed in the paper reviewed above are used to study the capillary motion of a fluid of very low viscosity rising on a vertical wall. It is claimed that the solutions obtained are in good agreement with reality.

A. M. Freudenthal (New York, N.Y.)

521:

Sentis, André. Sur le mouvement stationnaire en mécanique des milieux continus. C. R. Acad. Sci. Paris 245 (1957), 1694–1696.

The equation of motion of a linear visco-elastic medium described by a linear relation between the stress tensor and the strain tensor and its first two time-derivatives is solved for conditions of radial symmetry and under the assumption that the effect of the strain tensor and of its first derivative (elastic and viscous components) can be neglected in relation to the third term, an assumption that produces a simple second order differential equation for the velocity potential.

The physical meaning attributed to this solution in terms of a so-called potential of a "residual gravity force" in a medium subject to "decohesion" because of vanishing elastic and viscous forces seems to be unrelated to the initial assumption of a visco-elastic continuum. It should, moreover, be recalled that the specified visco-elastic equation itself has no counterpart in a physically possible visco-elastic mechanism.

A. M. Freudenthal (New York, N.Y.)

522:

Mandel, Jean. Sur les vibrations des corps viscoélastiques à comportement linéaire. C. R. Acad. Sci. Paris 245 (1957), 2176–2178.

This is one of a series of notes by the author on Laplace transform methods in linear visco-elasticity. Forced vibrations produced by the application of a time-dependent concentrated load are considered here. The author makes no reference to the large literature on the subject.

W. Noll (Pittsburgh, Pa.)

523:

Mikeladze, M. Š. Rigid-plastic analysis of anisotropic, almost cylindrical, shells. Soobšč. Akad. Nauk Gruzin. SSR 18 (1957), 265–270. (Russian)

The paper is concerned with the rigid-plastic behavior of cylindrical shells that carry no axial load. The shell material is anisotropic and obeys a yield law and flow rule which reduces to von Mises' law when the anisotropy is vanishingly small. The shell thickness is assumed to vary slightly in the axial direction.

Using the kinematical assumptions of the shell theory, the author derives the yield condition (in terms of the axial bending moment and circumferential membrane force) and the associated flow rule. Due to the variation of thickness in the axial direction, the yield condition depends on the axial coordinate.

The paper is essentially a generalization of the previous work on cylindrical shells [see, for example D. C. Drucker, Proc. 1st Midwest, Conf. Solids Mech., Urbana, Ill., 158–163 (1953)] and also contains some remarks about the theorems of limit design in the presence of the position dependent yield condition.

E. T. Onat (Providence, R.I.)

524:

Ford, Hugh; and Lianis, George. Plastic yielding of notched strips under conditions of plane stress. Z. Angew. Math. Phys. 8 (1957), 360–382.

Thin sheet strips with semi-circular and V notches pulled in tension and in bending in the plane of the sheet

are studied. Statically admissible stress fields consisting of uniform regions separated by discontinuities are presented to determine lower bounds on the loads, and velocity fields for upper bounds. For symmetrically placed notches in tension a lower bound is given to supplement Hill's upper bound. Single notches on one side of the strip only prescribe the bending problem. Close bounds are achieved. *E. H. Lee* (Providence, R.I.)

525:

Moskvitin, V. V. Static elasto-plastic deformations in the case of repeated cyclic loading. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 7, 24–32. (Russian)

526:

Lepik, Yu. R. On the equilibrium of flexible plates beyond the elastic limit. Prikl. Mat. Meh. **21** (1957), 833–842. (Russian)

This is a continuation of the author's earlier paper [Inženernyĭ Sbornik **24** (1956), 37–51] on large elastic-plastic deflections of plates of arbitrary strain-hardening material, using a method of elastic solutions proposed by Ilyushin. Two problems have been considered for clamped circular plates: (a) lateral axially symmetrical loading, the periphery of the plate being prevented from radial displacement; (b) lateral, and uniform radial loading. In the former case the Lagrange variational principle, in the latter case Galerkin's method have been used to expand unknown functions in power series. Numerical examples involving ideally elastic-plastic material have been computed by means of the digital computer Arrow.

J. Nowinski (Madison, Wis.)

527:

Grużewski, Aleksander. Application of a certain stochastic process to the computation of the mean geostatic pressure. Arch. Mech. Stos. **10** (1958), 115–125. (Polish and Russian summaries)

Under the trivial assumption that the mean weight of a layer of soil of certain depth below the surface is a linear function of the depth and specific gravities of the different geological formations making up this layer, combined with the assumption that in proceeding from the surface the probability of encountering boundaries between formations is governed by a Poisson distribution and is thus a "rare event", it is shown that the mean expected weight of the layer, although increasing with depth, is obviously not proportional to it.

The physical significance of the paper is somewhat obscure. *A. M. Freudenthal* (New York, N.Y.)

STRUCTURE OF MATTER

See also 638.

528:

Pirenne, Jean. Dynamics of a crystal lattice containing isotopes. Physica **24** (1958), 73–92.

By means of a moment method, the vibration spectrum of a monatomic crystal containing several isotopes, whose atoms are distributed at random throughout the lattice points, is obtained in terms of the solution of the conventional vibration problem of a single isotope crystal and in terms of the quantities

$$\langle \Delta\mu^p \rangle_{Av} = \sum_i c_i \Delta\mu_i{}^p = \sum_i c_i (\mu_i - \langle \mu \rangle_{Av})^p,$$

where c_i is the concentration of the ith isotope and $\Delta\mu_i$ the deviation of its inverse atomic mass μ_i from the average value $\langle \mu \rangle_{Av} = \sum_i c_i \mu_i$.

The frequency distribution is represented by a series development following the increasing powers of a heterogeneity parameter to which all the $\Delta\mu_i$ are assumed to be proportional. The first term is the frequency distribution of a fictitious crystal whose atoms would all have the mass $\langle \mu \rangle_{Av}^{-1}$. The next two terms are proportional to $\langle \Delta\mu^2 \rangle_{Av}$ and $\langle \Delta\mu^3 \rangle_{Av}$; the fourth term brings contributions in $\langle \Delta\mu^4 \rangle_{Av}$ and in $\langle \Delta\mu^2 \rangle^2$, etc. These three additional terms have been written down explicitly, but the following ones could be easily obtained.

Each term but the first one brings contributions of the form $\int \delta'(\lambda - \lambda') k(\lambda') d\lambda'$, where λ is the square of the angular frequency. These functionals may not be simply put equal to $K'(\lambda)$, as $K(\lambda)$ has singularities which must be carefully considered. In thermodynamic applications, however, this difficulty is irrelevant, for then the δ' function is simply replaced by the λ-derivative of the desired thermodynamic function of the harmonic oscillator and the remaining integral is trivial. *W. Nowacki* (Bern)

529:

Lifshitz, I. M.; and Kosevich, L. M. On the theory of the Shubnikov-de Haas effect. Soviet Physics. JETP **6** (1958), 67–77.

The oscillation of electrical conductivity in a magnetic field is a quantum effect and is investigated on the basis of formulas derived in a previous paper (I. M. Lifschitz, Soviet Physics, JETP **5** (1957), 1227–1234). For a magnetic field H in the z-direction, the oscillatory part of the conductivity tensor $\Delta\sigma$ is shown to be given by,

$$\Delta\sigma^{\alpha\beta} = H \sum_j \Big[\sum_k \frac{\partial}{\partial \zeta_0} (N_k{}^0 \overline{q_k{}^{\alpha\beta}})/\Sigma \frac{\partial N_k{}^0}{\partial \zeta_0} - q_{mj}{}^{\alpha\beta}(\zeta_0) \Big] \frac{d \ln S_{mj}}{d\zeta_0} \Delta M_j{}^z,$$

where $\alpha, \beta \neq z$, z, and

$$\Delta\sigma^{zz} = H \sum_j \Big[\sum_k \frac{\partial}{\partial \zeta_0} (N_k{}^0 \overline{q_k{}^{zz}})/\Sigma \frac{\partial N_k{}^0}{\partial \zeta_0} - q_{mj}{}^{zz}(\zeta_0) \Big] \frac{d \ln S_{mj}}{d\zeta_0} \Delta M_j{}^z$$

(if all the Fermi surfaces are convex) and

$$\Delta\sigma^{zz} = \frac{tr^3}{2\pi} H^2 \sum_i \chi_{0mi}{}^{zz}(\zeta_0) \Big(\frac{d \ln S_{mi}}{d\zeta_0} \Big)^2 \frac{\partial \Delta M_i{}^z}{\partial H}$$

(when there are Fermi surfaces with non-central cross-sections and the summation extends over all such Fermi surfaces).

The asymptotic behaviour of $\Delta\sigma^{\alpha\beta}$ in strong magnetic fields is next investigated and the oscillation amplitude is calculated in several simple cases.

Symbols: ζ_0 is the chemical potential for $H=0$; j, k indicate particular bands and the summations extend over all bands, q is the classical mobility tensor, S is the area of intersection of the constant energy surface for the electron with the plane p_z=const, perpendicular to the magnetic field and ΔM is the oscillatory part of the magnetic moment in the de Haas-van Alphen effect.

A. Raychaudhuri (Calcutta)

530:

Panichi, Ugo. Osservazioni di cristallografia armonica. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. II. (8) **5** (1957), 97–113.

Die Arbeit wendet die Theorie der harmonischen Punktequadrupel auf die Kristallographie an.

J. J. Burckhardt (Zürich)

FLUID MECHANICS, ACOUSTICS

See also 167, 169, 714, 761.

531:

Finn, Robert; and Noll, Walter. On the uniqueness and non-existence of Stokes flows. Arch. Rational Mech. Anal. **1** (1957), 97–106.

By a "Stokes flow" in a two- or three-dimensional domain, the authors mean a vector field $\mathbf{q}(P)$ and a scalar field $p(P)$, respectively three times and twice continuously differentiable and satisfying the equations $\Delta\mathbf{q}=\operatorname{grad} p$, $\operatorname{div}\mathbf{q}=0$; in a flow past an obstacle $\mathscr{B}$ (finite number of piecewise smooth, non-intersecting curves or surfaces), $\mathbf{q}$ is continuous in the exterior of $\mathscr{B}$ up to $\mathscr{B}$ and $\mathbf{q}=0$ on $\mathscr{B}$; p and the first derivatives of $\mathbf{q}$ are bounded up to $\mathscr{B}$; a flow is uniform at infinity if there exists a constant vector $\mathbf{q}_0$ (velocity at infinity) such that $|\mathbf{q}-\mathbf{q}_0|\to 0$ on every circle (or sphere) when the radius $r\to\infty$.

The main results of the paper are: (a) Uniqueness theorem: for any obstacle $\mathscr{B}$ there is at most one three-dimensional Stokes flow past $\mathscr{B}$, which is uniform at infinity and has a prescribed velocity at infinity; (b) non-existence theorem: for any obstacle $\mathscr{B}$, any bounded plane Stokes-flow past $\mathscr{B}$ is the state of rest $\mathbf{q}=0$.

The authors give several representation formulas for three-dimensional Stokes flow uniform at infinity, based on the theorem: let $\mathbf{q}$ be a Stokes-flow uniform at infinity; then, outside of a sphere of center Q and radius r, there is an harmonic vector field $\mathbf{u}$ (harmonic at infinity) such that $\mathbf{q}=\mathbf{q}+\mathbf{u}+\operatorname{grad}\psi$ where $\psi=-(\frac{1}{4})r^{\frac{1}{2}}\int_r^\infty s^{\frac{1}{2}}\operatorname{div}\mathbf{u}(Q+s\mathbf{e})ds$, $\mathbf{q}_0$ being the prescribed velocity at infinity and $\mathbf{e}$ a variable point on the unit sphere.

J. Kampé de Fériet (Lille)

532:

Fanti, Roy A.; Kemp, Nelson H.; and Nilson, Edwin N. A theory of thin airfoils, isolated and in cascade, yielding finite pressures at smooth leading edges. J. Aero. Sci. **25** (1958), 409–424.

This is a unified treatment of incompressible thin-airfoil theory, which is remarkable for giving the first solution for staggered cascades. The infinite leading-edge pressures of the classical theory are avoided by imposing the exact condition of tangent flow rather than its linearized form; the theory is approximate in that this condition is applied on the axis rather than at the actual surface. The resulting integral equations have been solved by iteration on an electronic computer, giving pressure distributions in good agreement with exact theory and experiment. The simpler results associated with the classical tangency condition are extracted, yielding explicit formulas. The reviewer notes that although the more complicated tangency condition avoids infinities it does not lead to stagnation points (e.g., for a parabola at zero incidence), so that the authors' hope of accurately predicting pressure peaks may be unjustified.

M. D. van Dyke (Paris)

533:

Rosenblat, S. The aerodynamic forces on an aerofoil in non-uniform unsteady motion in a closed tunnel. Philos. Trans. Roy. Soc. London. Ser. A. **250** (1957), 247–278.

The two-dimensional unsteady motion of a thin aerofoil, situated midway between parallel walls, and moving through an inviscid, incompressible fluid, is investigated. The physical plane is mapped into a rectangle by a transformation involving Jacobian elliptic functions; the theory is based on a solution of Laplace's equation satisfying given boundary conditions on this rectangle, and follows the analysis of Woods [Proc. Roy. Soc. London Ser. A, **229** (1955), 235–250; MR **16**, 972] for the problem of an unsteady aerofoil in a free jet.

The analysis is given for a general distribution of upwash, and the special cases in which the upwash is (a) a sudden upgust and (b) a harmonic oscillation are treated in detail. The formulae derived for non-dimensional air-load coefficients are at variance with corresponding results obtained by Timman [Appl. Sci. Research A. **3** (1951); 31–57; MR **14**, 595] and Lilley [Coll. Aero. Cranfield. Rep. no. 60, (1952)]; this is attributed to the latter authors having performed invalid operations on the Fourier series used in their methods of solution.

D. A. Spence (Farnborough)

534:

Mallick, D. Motion induced by a sphere vibrating along the axis of rotation of an infinite rotating liquid. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 544–559.

Le mouvement engendré dans un liquide en rotation par une sphère se déplaçant suivant l'axe de rotation a fait l'objet de travaux de G. I. Taylor [Proc. Roy. Soc. London Ser. A **102** (1922), 180–189], R. R. Long [J. Meteorology **10** (1953), 197–203], K. Stewartson [Proc. Cambridge Philos. Soc. **48** (1952), 168–177; MR **13**, 997], qui ont mis en évidence certaines particularités concernant l'indétermination de ces mouvements, et l'existence possible de surfaces de discontinuité. Dans le présent travail la sphère est animé, à partir de l'instant $t=0$, d'un mouvement oscillatoire $U=V_0\cos\beta t$ suivant l'axe de rotation du fluide, supposé infini et parfait. Les mouvements causés par la sphère sont supposés petits, et les équations sont linéarisées. Utilisant la transformation de Laplace, les solutions sont données sous forme d'intégrales, et leur valeur asymptotique, pour t tendant vers l'infini, est étudiée.

L'expression de la vitesse sur l'axe et sur le cylindre enveloppe de la sphère est donné pour ce mouvement limite.

Le mouvement dépend du paramètre $\lambda=2\Omega/\beta$ (Ω étant la vitesse angulaire de rotation), la valeur $\lambda=1$ étant singulière, et il existe deux cônes, centrés sur l'axe, tangents à la sphère, qui sont des surfaces singulières pour le champ des vitesses.

R. Gerber (Grenoble)

535:

Lambert, John W. On the nonlinearities of fluid flow in nonrigid tubes. J. Franklin Inst. **266** (1958), 83–102.

Angeregt durch die medizinische Frage nach der Bewegung des Blutes in den Adern wird die Strömung in einem glatten elastischen Rohr untersucht. Unter der Voraussetzung einer reibungslosen, inkompressiblen Flüssigkeit lässt sich bei der Beschränkung auf eindimensionale Bewegung das Problem durch zwei gekoppelte nichtlineare Differentialgleichungen beschreiben. Für den Fall des linearen Zusammenhanges zwischen dem Druck der Flüssigkeit und dem Rohrradius wird eine allgemeine Lösung angegeben, die zur Auswertung mit digitalen Rechenmaschinen geeignet ist. Beispiele, die der Blutströmung in der Aorta entsprechen, sind durchgerechnet und mit Messungen verglichen worden.

L. Speidel (Mülheim)

536:

Fraenkel, L. E. Incompressible flow past quasi-cylindrical bodies and some associated problems. Quart. J. Mech. Appl. Math. **11** (1958), 212–222.

It is pointed out that the solution of the potential

problem corresponding to axisymmetric incompressible perfect-fluid flow past a nearly cylindrical solid body can be written down immediately in terms of the cylindrical harmonic

$$G_1(x,r)\equiv\frac{1}{\pi}\int_0^\infty\frac{\cos(\omega x)K_0(\omega r)}{\omega K_1(\omega)}\,d\omega\quad(r\geqq 1)$$

and its partial derivative $G(x,r)\equiv\partial G_1/\partial x$. K_0 and K_1 are the modified Bessel functions. First the asymptotic expansions of these functions are found and then their numerical evaluation is considered. Limited numerical values are tabulated. It is shown how these functions can be used to construct the flow past slender bodies of revolution with discontinuous profile slope, and as an example the pressure distribution on a cone-cylinder is calculated.

W. R. Sears (Ithaca, N.Y.)

537:

Nickel, Karl. Das Unabbängigkeitsprinzip bei dreidimensionalen Grenzschichten. Arch. Math. **9** (1958), 313–320.

In special cases the solution of the boundary layer equations in three dimensions can be reduced to a two dimensional problem (e.g., boundary layer on bodies of revolution, on swept back wings). The author studies this problem on a more general basis. First, the system formed by the three Navier-Stokes equations for incompressible flow and the continuity equation is considered. If the velocity vector and the pressure do not depend on the coordinate y, the solution of the system is reduced to solving first a system of three coupled equations (two N.-S. equations and the continuity equation). The third equation is linear in the unknown v and thus can be solved easily. The boundary layer equations are obtained from the N.-S. equations by neglecting small terms, which leads to assuming the pressure to be independent of the coordinate z normal to the wall and to be known from inviscid considerations or experiments. With this assumption the system is reduced to three equations. Instead of using rectilinear coordinates x, y, z, the author introduces general curvilinear coordinates, which promise a better interpretation of the results for very general curved surfaces. Criteria are established for the system of three equations to be decomposed into a system of two coupled differential equations and one differential equation, which contains only one unknown after the system of two equations is solved. These criteria (the principle of independence) include the known cases. The author discusses the relation between his principle and the prevalence or predominance principle of Eichelbrenner-Oudart. Finally, it is mentioned that this "independence principle" does not include all possible decompositions of the system of boundary layer equations. The author says that his investigation can be generalized to a similar study of the compressible laminar boundary layer.

I. Flügge-Lotz (Stanford, Calif.)

538:

Phillips, O. M. The equilibrium range in the spectrum of wind-generated waves. J. Fluid Mech. **4** (1958), 426–434.

It is suggested that the smaller-scale components of wind-generated waves may be in a condition of statistical equilibrium when the duration and fetch of the wind are large. Dimensional arguments are given to show that for large values of the frequency ω, the frequency spectrum $\Phi(\omega)$ is of the form

$$\Phi(\omega)\sim\alpha g^2\omega^{-5},$$

where α is an absolute constant. (Reasons are given why the wind velocity should not occur in this formula.) The instantaneous spatial spectrum is proportional to k^{-4} for large wave numbers k; this is consistent with the occurrence of sharp crests. The loss of energy per unit surface area per unit time from the waves to turbulence and heat is proportional to $\rho_w u_*^3$, where ρ_w is the water density, and u_* is the friction velocity at the surface.

The prediction of the form of $\Phi(\omega)$ agrees well with measurements made by Burling. It does not agree with the form proposed by G. Neumann, who was, however, not concerned with this range of frequency.

F. Ursell (Cambridge, England)

539:

***Kaplan, Paul. The waves generated by the forward motion of oscillatory pressure distributions.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 316–329. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

The transient linearized wave motion of an incompressible fluid of infinite depth due to a translatory surface pressure disturbance which is periodic in time are given. The "steady-state" motion is then obtained — in the fashion of J. J. Stoker, T. Y. Wu, and the reviewer — directly as a limit of the transient motion as the time $t\to\infty$.

C. R. DePrima (Pasadena, Calif.)

540:

Crease, J. The propagation of long waves into a semi-infinite channel in a rotating system. J. Fluid Mech. **4** (1958), 306–320.

In an effort to explain the behaviour of tides and storm surges in the North Atlantic, the author considers the propagation of long waves into a canal which is rotating at constant angular velocity. The depth of the fluid is uniform. In particular, the propagation into a canal formed by two semi-infinite barriers as well as that into a canal formed by one infinite and one semi-infinite barrier is considered. This research represents a continuation of the author's previous work [same J. **1** (1956), 86–96; MR **18**, 776] and the effects of the additional barriers are discussed. The problem is formulated as a "double" or "mixed" Wiener-Hopf problem which conveniently separates into two uncoupled integral equations. The observational data for the North Sea is compared with the predictions of each model and is found to be between the two cases.

H. Greenspan (Cambridge, Mass.)

541:

Voĭt, S. S. Waves on the surface of a liquid arising from a variable system of pressures. Trudy Morsk. Gidrofiz. Inst. **10** (1957), 3-9. (Rússian)

A pressure $p=f(x,t)$ (an even function of x) is applied to a liquid surface over a region growing with the velocity c. The potential Φ of a slow motion is found by using the conditions

$$\frac{1}{g}\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial z}=\frac{1}{\rho g}\frac{\partial f}{\partial t},\quad z=0,$$

$$\Phi=0,\ \frac{\partial\Phi}{\partial t}=0,\quad t=0.$$

In the special case when the pressure is uniform over the region, the elevation ξ of the surface may be reduced to

$$\xi=A(\gamma)\cdot\frac{1}{(gt/c)^{\frac{1}{2}}}\cos\left[\frac{g}{4v^2}(vt-\alpha)+B(\gamma)\right],$$

where $(gt/c)^{-\frac{1}{3}}A(\gamma)$ is an oscillating function, v is the constant velocity of an observer, and $\gamma=c/2v$. The observed wave-length is $8\pi v^2/g$, and the period is $8\pi v/g$.
C. D. Calsoyas (Livermore, Calif.)

542:

Aris, R. On the dispersion of linear kinematic waves. Proc. Roy. Soc. London. Ser. A. **245** (1958), 268–277.

The term 'kinematic wave' was introduced by Lighthill and Whitham [Proc. Roy. Soc. London Ser. A **229** (1955), 281–316; MR **17**, 309] for wave motions defined by $\partial k/\partial t+\partial q/\partial x=0$, $q=f(k)$. When the second relation is replaced by a more accurate one involving also the derivatives of q and k the waves diffuse. If the relations are linear (so that $f(k)=ak+b$ in the first approximation) the author notes that it is relatively easy to find the growth of the mean and the variance for a hump-like disturbance. He shows that the mean moves with the kinematic wave velocity $f'(k)=a$ and the increase in the variance is proportional to the distance travelled. These results are valid for all times whereas in the paper cited above the result that the maximum disturbance moves with velocity $f'(k)$ was limited to sufficiently large times. It should be noted, however, that the velocity of the mean does not in general give the velocity of the "main disturbance" (which is more appropriately represented by the maximum or the median) for the earlier times.

The method is applied to a problem of a temperature wave in the flow of a fluid through a packed bed, where the subsidiary relations between q, k and their derivatives are a complicated set of linear equations. The mean and the variance are quickly found even though the full solution would be intractable.
G. B. Whitham (New York, N.Y.)

543:

Yamada, Hikoji. On approximate expressions of solitary wave. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **6** (1958), 35–47.

The experiments at MIT of Daily and Stephan have shown good agreement with the approximate solitary-wave theory of Boussinesq, but less good agreement with the theories of Rayleigh, McCowan, and Weinstein, some of which carry the approximation to higher terms. In the present paper the logarithm of the complex velocity of the solitary wave is expanded in a power series in a complex variable which is related in a simple manner to the complex potential. The coefficient in the power series are to be found from a pair of non-linear integral equations. An approximate method of solution is described, and applied to the solitary wave for which the height-depth ratio is 0.4808. This is found to agree very closely with Boussinesq's formula, less closely with the other theories. A comparison is also made with the explicit formula of Packlam where the exact non-linear boundary condition is replaced by T. V. Davies' approximate non-linear boundary condition. *F. Ursell* (Cambridge, England)

544:

***Tipei, N. Hidro-aerodinamica lubrificatiei. [Hydro-aerodynamics of lubrication.]** Biblioteca Ştiinţelor Tehnice, I. Editura Academiei Republicii Populare Romîne, 1957. 695 pp. (1 insert) Lei 37.00.

The present book seems to be the most extensive treatise concerning modern lubrication theory. Also, the book contains a wealth of information about the technical applications of the various theoretical findings, and, where possible the computational results are compared with experimental data.

Furthermore, the author treats lubrication theory in a very general manner. As a matter of fact, the first three chapters (and also some parts of chapter IV) are primarily devoted to the various fundamental notions related to, e.g., the motion of viscous fluids, density and viscosity variations, thermal effects and the estimation of the order of magnitude of the various terms in the non-linear partial differential equations.

Chapter IV is devoted to a study of bearings subject to constant forces and velocities. Typical topics: pressure distribution in journal bearings of infinite elongation, the application of power series, Sommerfeld and Korovcinskiĭ's complex-functional treatment, variational and finite-difference methods for three-dimensional problems. Chapter V is primarily devoted to bearings with no radial clearance, pressure determination and global characteristics of bearings with no radial clearance. In chapter VI, the author presents the difficult theory of bearings with variable geometric configurations. Typical topics: pressure determination by means of finite-difference methods, constant and variable clearance with constant and variable radii, global characteristics of bearings with variable elements, plane surfaces with first and second-order discontinuities, lemon bearings, hydrostatic lubrication, lubrication of spherical surfaces, etc.

Chapter VII is devoted to bearings subject to variable forces and velocities. Typical topics: plane surfaces of infinite and finite elongation, circular cylindrical surfaces of infinite and finite elongation.

In chapter VIII, the author presents results related to the important problem of the stability of motion of lubricated bodies. Typical topics: Plane and circular cylindrical surfaces, centrifugal and constant loads, approximate methods, etc. Chapter IX is devoted to a more general consideration of hydrodynamic lubrication. Typical topics: general power-series solutions (subject to specified hypotheses), dependence of viscosity upon pressure, viscous fluid motion in thick layers, lubrication of rolling circular cylindrical surfaces, boundary-value problems, rates of discharge, etc.

Chapter X is devoted to gaseous lubrication. The method of presentation is similar to those of the preceding ones.

In the present book, the author has made a serious attempt to reduce the gap between theory and experiment in lubrication theory. Indeed, this is by no means a simple matter, since the equations to be solved present formidable mathematical difficulties. As a matter of fact, the equations are just as complicated (if not more) as the well-known Navier-Stokes equations of motion of non-linear hydrodynamics. Hence, rigorous solutions of these systems of equations are not feasible at present. These mathematical difficulties are usually circumvented by the omission of various terms in the fundamental equations of motion so as to make the ensuing equations more amenable to approximations. The author enumerates a variety of equations (with varying degrees of accuracy) related to lubrication phenomena. This is an excellent feature of the present book. References are made to a large number of papers and books. However, references to some important works of Leĭbenzon and Loĭtyanski are lacking.

Unfortunately, there are many printing errors to be found in this book; some of them also appear in the lists of references. *K. Bhagwandin* (Oslo)

545:

Hasimoto, Hidenori. On the flow of a viscous fluid past a thin screen at small Reynolds numbers. J. Phys. Soc. Japan **13** (1958), 633–639.

The slow steady flow of a viscous incompressible fluid past a thin screen with holes or slits is investigated on the basis of the Stokes approximation. This boundary value problem is solved by a harmonic function similar to the stamp problem in the elasticity. As examples, the cases of a single elliptical hole, a single slit, two parallel and equal slits, and a series of parallel and equal and equidistant slits are discussed. *L. N. Tao* (Chicago, Ill.)

546:

Brenner, Howard; and Happel, John. Slow viscous flow past a sphere in a cylindrical tube. J. Fluid Mech. **4** (1958), 195–213.

A viscous liquid is flowing steadily and slowly through a cylindrical container of radius R_0 and has a small sphere of radius a immersed in it at an arbitrary distance from the cylinder axis. The sphere is supposed to be moving parallel to the walls with constant speed. The Stokes equation is used to derive the first two terms of expansions in powers of a/R_0 for the drag, rotational couple and pressure drop by the superposition of 'image' velocity fields.

The reviewer believes that there may be some trouble here since the Stokes equation does not yield a valid description of the perturbation velocities at large values of r/a. *H. C. Levey* (Nedlands)

547:

Sakurai, Akira. Three-dimensional steady, radial flow of viscous, heat-conducting, compressible fluid. Quart. J. Mech. Appl. Math. **11** (1958), 274–289.

If an inviscid gas with zero thermal conductivity existed, it could flow steadily and radially only exterior to the sphere at which sonic speed is attained and, moreover, the speed in this region would be two-valued, being either subsonic or supersonic. The flow of a real gas in the limit of vanishing viscosity and thermal conductivity should not have this character and, in fact, this is a singular perturbation problem. In the two-dimensional case [Levey, Quart. Appl. Math. **12** (1954), 25–48; MR **16**, 190], the main solutions consist of shocks of finite thickness joining parts which lie close to inviscid solutions.

A Reynolds number is defined for the three dimensional problem and its reciprocal is the perturbation parameter. Suitable assumptions about the Prandtl number allow the governing equations to be reduced to a single second order ordinary differential equation, and using the two-dimensional result as a guide, approximate analytical solutions are obtained in appropriate sub-regions of the flow and patched together by "boundary layer" techniques. *H. C. Levey* (Nedlands)

548:

Emslie, Alfred G.; Bonner, Francis T.; and Peck, Leslie G. Flow of a viscous liquid on a rotating disk. J. Appl. Phys. **29** (1958), 858–862.

This paper deals with the spreading of a thin layer of viscous fluid over the surface of a rapidly spinning disc by centrifugal action. The motion relative to the disc is assumed slow and the usual boundary layer assumptions lead to a solution for the film thickness in terms of its initial distribution.

Examples show that the layer quickly takes the form of a film of uniform thickness spreading outwards, but, if the initial negative surface slope is too great, the "wave-front" ultimately becomes reentrant and the thickness triple-valued. Before this stage is reached the original assumptions are violated. *H. C. Levey* (Nedlands)

549:

Oudart, Adalbert. Couche limite tridimensionnelle: lame hélicoïdale en viration uniforme dans un fluide au repos; fluide en viration uniforme sur pale hélicoïdale fixe. Z. Angew. Math. Phys. **9b** (1958), 479–489.

The flow of viscous fluids moving past a fixed thin helical plate with a constant velocity parallel to the axis of the plate and an angular rotation is treated by the method of boundary-layer approximation. The boundary layer consists of a primary flow of the Blasius type and a secondary radial flow. The computation is effected by the integral method. The case in which the plate moves and the fluid is at rest has also been considered. *Y. H. Kuo* (Peking)

550:

Payne, R. B. Calculations of unsteady viscous flow past a circular cylinder. J. Fluid Mech. **4** (1958), 81–86.

A numerical solution has been obtained for the starting flow of a viscous fluid past a circular cylinder at Reynolds numbers 40 and 100. The method used is the step-by-step forward integration in time of Helmholtz' vorticity equation. The advantage of working with the vorticity is that the calculations can be confined to the region of non-zero vorticity near the cylinder.

The general features of the flow, including the formation of the eddies attached to the rear of the cylinder, have been determined, and the drag has been calculated. At $R=40$, the drag on the cylinder decreases with time to a value very near that for the steady flow. (Author's summary.) *M. Lister* (State College, Pa.)

551:

Miyagi, Tosio. Viscous flow at low Reynolds numbers past an infinite row of equal circular cylinders. J. Phys. Soc. Japan **13** (1958), 493–496.

552:

Kotlyar, Ya. M. Some examples of motion of a viscous gas in a narrow channel of variable width. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 5, 34–39. (Russian)

553:

***Eichelbrenner, E. A. Décollement laminaire en trois dimensions sur un obstacle fini.** Office National d'Etudes et de Recherches Aéronautiques, Publ. No. 89, 37 pp. (1957).

This is a continuation of ONERA work on questions of boundary-layer separation on three-dimensional bodies. It is pointed out that the Sears-Oudart criterion [Eichelbrenner and Oudart, same Publ. No. 76 (1955); MR **17**, 425] is satisfied at every limiting streamline (line of flow direction at the surface), and therefore does not identify uniquely the separation line. Other criteria and methods of calculation are described briefly. An improved criterion of R. Legendre [Rech. Aéro. no. 54 (1956), 3–8; MR **18**, 696] is stated and is applied here to flow over a flat (1:3:6) ellipsoid. The calculation of boundary-layer flow was carried out in coordinates defined by the potential flow, using a Mangler-type transformation and a Pohlhausen procedure. The principal results are five-color

diagrams showing separation lines and several related curves as well as points of convergence and divergence on the surface of the ellipsoid at various angles of incidence. A significant change of flow pattern occurs at a critical angle of incidence; viz., a configuration involving, on the upper surface, two points of divergence and a central point of convergence is replaced by one having a central point of divergence only.

Experiments have been carried out in flow of water past ellipsoids, using milky filaments for flow visualization. There is substantial qualitative agreement between observation and theory. In particular, at a critical angle of incidence there is a change from an unstable, asymmetric type of separation to a stable, symmetric one on the upper surface. *W. R. Sears* (Ithaca, N.Y.)

554:

Schultz-Grunow, F. Zur Entstehung von Längswirbeln in Grenzschichten. Z. Angew. Math. Mech. **38** (1958), 85–95. (English, French and Russian summaries)

This is an experimental and theoretical study of the generation of longitudinal vortices in the boundary layer of a flat plate. It is pointed out that trailing vortices in the sense of wing theory are produced by shear flow over symmetrical obstacles, and it is proposed that this mechanism occurs when boundary-layer flow impinges on an element of surface roughness. Reference is made to theoretical studies of shear flow by Taylor [Proc. Roy. Soc. London. Ser. A. **92** (1916), 408–424], von Kármán and Tsien [Quart. Appl. Math. **3** (1945), 1–11; MR **6**, 248], and others. Experimental results by Schlichting [Ing.-Arch. **7** (1936), 1–34], showing the velocity field behind roughness elements in a turbulent boundary layer are appealed to, as well as more recent experimental results. Flow pictures obtained by the author, using a new technique of flow visualization [Schultz-Grunow and Hein, Z. Flugwiss. **4** (1956), 28–30], are presented. These show longitudinal vortices trailing behind roughness elements in boundary layers at various Reynolds numbers in both steady and transient situations. In another series of tests an oscillating wing was used to superimpose transverse vortices upon the longitudinal vortices of an array of roughness elements. From the flow pictures the author concludes that the longitudinal vortices do not affect the Tollmien stability diagram. Finally, another source of longitudinal vortices is mentioned and pictured, viz. the appearance of vortices on the front side of a cylinder in a shear flow, which extend back into the flow behind the cylinder. To this reviewer these would seem to be just the "secondary-flow" vortices predicted by Hawthorne [J. Aeronaut. Sci. **21** (1954), 588–608, 648; MR **16**, 190] and others. *W. R. Sears* (Ithaca, N.Y.)

555:

Mitchell, Andrew R.; and Thomson, John Y. Finite difference methods of solution of the von Mises boundary layer equation with special reference to conditions near a singularity. Z. Angew. Math. Phys. **9** (1958), 26–37.

The authors consider the numerical solution of von Mises' equation for the incompressible boundary layer on a semi-infinite flat plate with adverse pressure gradient:

$$(*) \qquad z_x = u z_{\psi\psi},$$

where x is distance along the plate, ψ the stream function, u the velocity in the boundary layer, $u_1 = 1 - x$ the velocity outside the boundary layer, and $z = u_1^2 - u^2$. The authors solve (*) by finite difference methods. There is unavoidable (but slight) instability in the difference scheme near the plate ($\psi = 0$). Values of z, u in the unstable region are improved using series expansions in powers of $\psi^{\frac{1}{2}}$. Results using the combined methods agree very well with results obtained using more cumbersome techniques. Previously it was thought that a finite difference approach could not give good results because $z_{\psi\psi\psi} = \infty$ at the plate. *M. A. Hyman* (Yorktown Heights, N.Y.)

556:

Watson, J. A solution of the Navier-Stokes equations illustrating the response of a laminar boundary layer to a given change in the external stream velocity. Quart. J. Mech. Appl. Math. **11** (1958), 302–325.

Exact solutions of the Navier-Stokes equations are derived by a Laplace-transform technique for two-dimensional incompressible flow past an infinite plane porous wall. It is assumed that the flow is independent of the distance parallel to the wall and that the velocity component normal to the wall is constant. A general formula is derived for the velocity distribution as a function of the given free-stream velocity and, from this, general formulae for the skin friction and displacement thickness are obtained. Among the particular cases considered are (i) flow in which the free-stream velocity is changed impulsively from one value to another, (ii) uniform acceleration of the main stream from a given value of the velocity, and (iii) flow in which the free-stream velocity consists of a mean value with a superimposed decaying oscillation. In each case, detailed analytical results illustrating the flow characteristics are obtained. (From author's summary.) *D. W. Dunn* (Ottawa, Ont.)

557:

Sowerby, L. An approximation to the boundary layer flow along an edge. Quart. J. Mech. Appl. Math. **11** (1958), 290–301.

The incompressible laminar boundary layer flow along an infinite flat quarter plate is studied, one edge of the plate being parallel to the main stream, the other normal to it. Parabolic coordinates are used in planes parallel to the leading edge. Special attention is given to the computation of boundary layer thickness and shear stress near the edge of the plate which is parallel to the main stream. The computation is done by using series expansions for the velocity components and establishing ordinary differential equations for the coefficients.

I. Flügge-Lotz (Stanford, Calif.)

558:

Baxter, Donald C.; and Flügge-Lotz, Irmgard. Compressible laminar boundary layer behavior studied by a finite difference method. Z. Angew. Math. Phys. **9b** (1958), 81–96.

A method of computing the skin friction and heat transfer in laminar compressible flow along curved surfaces is discused. The system of nonlinear partial differential equations which describe the flow in the thin layer near a surface in which rapid temperature and velocity changes occur, are written in Crocco's form. The finite difference schemes which are used to obtain an approximate numerical solution are given, together with criteria for the stability and convergence of the procedure. A program, written for the I.B.M. 650 digital computer, was used to ccmpute solutions to some 60 examples. The results are recorded and discussed.

M. Lister (State College, Pa.)

559:

***Sandborn, Virgil A. An approximate equation for the mean velocity distribution in an incompressible turbulent boundary layer.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 85–107. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

The expression

$$u/u_1=1+\zeta(1-y/\delta)^2-(1+\zeta)(1-y/\delta)^{2n},$$

where $\zeta=(s-n)/(n-1)$ and $2s=[\partial(u/u_1)/\partial(y/\delta)]_{y/\delta=0}$, is proposed as a means of representing the velocity profiles $(u/u_1)(y/\delta)$ of incompressible turbulent boundary layers. This expression is shown to approximate to the expected form $u/u_\tau=f(yu_\tau/\nu)$ (u_τ=friction velocity, ν=kinematic viscosity) in the core of the layer and, by suitable choice of the form parameter n, to fit well profiles observed in zero and moderately adverse pressure gradients. Close to a separation point it becomes less satisfactory, and despite its simplicity, it is for this reason perhaps less likely to lead to an advance in our understanding of the turbulent boundary layer than is the "law of the wake" proposed by Coles [J. Fluid Mech, **1** (1956), 191–226; MR **18**, 355].

D. A. Spence (Farnborough)

560:

Szablewski, W. Zur Theorie der laminaren Unterschicht turbulenter Grenzschichtströmungen. Z. Angew. Math. Mech. **38** (1958), 77–80.

In view of the controversy at present surrounding the existence of a laminar sub-layer, the author has used the full Navier-Stokes equations to estimate the intensities of the fluctuating velocity components in stationary turbulence in the immediate vicinity of a wall on which there is a two dimensional turbulent boundary layer. He finds, in agreement with measurements by Laufer [NACA Tech. Notes **TN 2934** (1953)], that the longitudinal and transverse components vary as the distance y from the wall, while the normal component varies as y^2, solong as the Reynolds number based on y and the friction velocity is less than 10. In this region also, the turbulent shear stress varies as y^3, and is thus negligible in comparison with the viscous stress. The existence is therefore inferred of a quasi-laminar sublayer in which momentum and heat are transferred by molecular processes, as distinct from the eddy processes which operate in the remainder of the boundary layer. The energy balance of turbulence in the sub-layer is also examined.

D. A. Spence (Farnborough)

561:

Bertotti, Bruno. Sul flusso secondario nello strato limite tridimensionale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 455–458.

Secondary flow is that in which the velocity in the boundary layer possesses a component normal to the streamline at the outer edge. It is shown for a general surface that such a component will be present in all cases in which the external streamlines, when projected orthogonally on the surface, do not follow geodesics. This has also been demonstrated, in the converse form, by L. C. Squire [Aero. Res. Council. Rep. and Memo. no. 3006 (1955); MR **19**, 88]. The author shows further, that the secondary flow velocity is small compared with that of the free stream, provided the boundary layer thickness is small compared with the Gaussian radius of curvature of the surface.

D. A. Spence (Farnborough)

562:

Hammitt, Andrew G. The interaction of shock waves and turbulent boundary layers. J. Aero. Sci. **25** (1958), 345–356.

On the basis that the mass and momentum of a boundary layer are conserved through the region of interaction with a shock wave, a theory is presented for predicting the overall changes caused by such interaction. The theory is presented in terms of the Crocco-Lees parameters f and κ. For weak interactions, equations analogous to the usual shock wave equations are obtained. For interactions strong enough to cause flow separation, the theory is developed in terms of a distributed pressure rise rather than a discontinuous jump. Edge conditions are discarded in favour of an empirical $f-\kappa$ relationship based on studies of equilibrium flows.

Experimental results for weak interactions are in good agreement with the theory. For strong interactions the case made is less convincing: possibly a combination of the two methods would more adequately meet the situation.

J. Seddon (Farnborough)

563:

Fay, J. A.; and Riddell, F. R. Theory of stagnation point heat transfer in dissociated air. J. Aero. Sci. **25** (1958), 73–85, 121. .

Effects of diffusion and atom recombination are included in a general development of the boundary layer equations for the case of very high speed flight where the external flow is in a dissociated condition. For the stagnation point boundary layer, a similarity transformation is given which reduces the problem to a set of ordinary differential equations. Numerical solutions were obtained on a digital computer for thermodynamic equilibrium, "frozen" flow, and for a finite recombination rate. The results for equilibrium heat transfer are correlated by a simple formula, and it is shown that lack of equilibrium in the boundary layer makes very little difference, provided the wall does not inhibit atom recombination and the Lewis number is near unity. But if the wall is noncatalytic, then in the "frozen" case the heat transfer may be much reduced.

The accuracy of estimates based on the solutions is restricted not so much by the simplifying assumptions in the analysis as by lack of knowledge of physical quantities such as viscosity and Lewis number.

L. F. Crabtree (Farnborough)

564:

Lunev, V. V. Application of the small perturbation method to laminar boundary layer problems. Prikl. Mat. Meh. **21** (1957), 606–614. (Russian)

The equation of the laminar boundary layer flow of a compressible gas with small gradient pressure, with arbitrary dependence of viscosity on temperature, and with variable Prandtl number is solved. Dissociated air does not change the method of solution. The results apply to supersonic laminar boundary layers on thin bodies, bodies that differ slightly from a wedge or cone, and so forth. (Author's summary.)

J. P. LaSalle (Baltimore, Md.)

565:

Hansen, A. G. Possible similarity solutions of the laminar, incompressible, boundary-layer equations. Trans. A.S.M.E. **80** (1958), 1553–1559; discussion, 1559–1562.

Review article.

566:

Chandrasekhar, S. The stability of viscous flow between rotating cylinders. Proc. Roy. Soc. London. Ser. A. **246** (1958), 301–311.

In most of the previous work on this problem it has been assumed that the difference in the radii of the two cylinders is small compared to their mean radius. When this assumption is not made the problem is more difficult and the present paper describes a method for dealing with this more general situation. The method is based upon an expansion in terms of a particular set of orthogonal functions (defined by a certain linear combination of the four Bessel functions of order one) which automatically satisfy four of the boundary conditions. Detailed results are obtained for the critical Taylor number and wave number at the onset of instability for the case in which $R_1/R_2=\frac{1}{2}$ and $-0.5\leqq\Omega_2/\Omega_1\leqq+0.25$, where R_1 and R_2 are the radii of the inner and outer cylinders, respectively, and Ω_1 and Ω_2 are their angular velocities. The value $\Omega_2/\Omega_1=0.25$ corresponds to Rayleigh's criterion for an inviscid fluid. *W. H. Reid* (Providence, R.I.)

567:

Tatsumi, T.; and Kakutani, T. The stability of a two-dimensional laminar jet. J. Fluid Mech. **4** (1958), 261–275.

Stability with respect to the antisymmetric disturbance is considered. The curve of neutral stability is obtained by developing the solution in powers of $(\alpha R)^{-1}$ for the upper branch of the curve and in powers of αR for the lower branch, where α is the wave-number and R the Reynolds number. The asymptotic behaviour is $\alpha\to 2$, $C\to 2/3$ for $R\to\infty$ on the upper branch, and $R\sim 1.12\alpha^{-\frac{1}{2}}$, $C\sim 1.20\alpha^2$ for $\alpha\to 0$ on the lower branch. The entire neutral curve is not obtained because of slow convergence of the series, but the branch for $\alpha\to 0$ is sufficient to determine a minimum critical Reynolds number $R=4.0$ at $\alpha=0.2$. Because of the occurrence of small values of αR, some discussion is given on the validity of the basic assumptions of the stability theory of nearly-parallel flows.

D. W. Dunn (Ottawa, Ont.)

568:

✱Lessen, M. On the hydrodynamic stability of curved laminar compressible flows. Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 22–28. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

The author considers the disturbance equations for compressible inviscid flow in cylindrical polar coordinates, and after reducing them to a single equation examines an approximate version of this equation (for which a detailed derivation is not given). Since the terms depending on the velocity gradient and entropy gradient have opposite effects on the stability characteristics, it is concluded that instability may occur in the flow about bodies at high speed under certain circumstances.

D. W. Dunn (Ottawa, Ont.)

569:

Sexl, Theodor; und Spielberg, Kurt. Zum Stabilitätsproblem der Poiseuille-Strömung. Acta Phys. Austriaca **12** (1958), 9–28.

The stability of the laminar flow in a circular tube is considered. The general method of approach used previously by Sexl [Ann. Physik **83** (1927), 835–848; **84** (1927), 807–822] is retained, but a more exact investigation is made of the solution of the differential equation for the disturbance amplitude function $\Phi(r)$. The proof of stability with respect to small two-dimensional disturbances is carried out by examining separately the two cases of small and large amplification factor c_i, and is based on two integral conditions derived from the differential equation and boundary conditions.

D. W. Dunn (Ottawa, Ont.)

570:

✱Björgum, Oddvar. On the possibility of a mathematical theory of shear-flow turbulence. Corso sulla teoria della turbolenza, Vol. 1, pp. 123–126. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto e Bella, Turin, 1957. viii+339 pp.

This note considers the possibility of developing a theory of turbulence that will predict correctly the mean velocity distribution in a given flow, in particular, in the Poiseuille flow between parallel planes. As a first approximation, the statistically steady turbulent motion might be represented as the superposition of neutrally stable solutions of the Orr-Sommerfeld equation, distributed so that they produce the Reynolds stresses appropriate to the mean velocity distribution. The problem of determining the distribution is not determinate, but it might be made determinate either by using higher-order approximations involving the non-linear effects or possibly by using another type of approximation.

A. A. Townsend (Cambridge, England)

571:

✱Batchelor, G. K. The singularity in the spectrum of homogeneous turbulence. Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 67–72. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

The author summarizes some of the results expounded in a joint paper with I. Proudman [Philos. Trans. Roy. Soc. London Ser. A. **248** (1956), 369–405; MR **18**, 843]; the main result is that, if one takes account of all the requirements, in the case of an homogeneous turbulence, the energy spectrum must have a singularity of a certain kind at the origin in wavenumber space. It has been overlooked in the past that the fact that the velocity field is governed by the Navier-Stokes equations does not permit the tensor covariance $R_{ij}(\mathbf{r})$ to decrease as rapidly as $|\mathbf{r}|\to\infty$ so that all integral moments converge. This fact bears important consequences to some assumptions (e.g., Loitzansky's invariant) made some years ago in the theory of turbulences; a model of turbulence in which the spectrum is analytic near zero wave-number appears as inadequate. *J. Kampé de Fériet* (Lille)

572:

Batchelor, G. K. Diffusion in free turbulent shear flows. J. Fluid Mech. **3** (1957), 67–80.

The paper is concerned with the statistical properties of the displacement of a marked fluid particle released from a given point in a turbulent shear flow generated at a point (jet or wake), having a similar structure at different stations downstream; it is supposed that at a distance x downstream, the length and velocity scales, $L(x)$, $V(x)$ are proportional to x^p and x^{-q} (e.g., for a round jet $p=q=1$, for a plane jet $p=1$, $q=\frac{1}{2}$, etc.). A particle is carried downstream and the statistical properties of its velocity $u(t)$ change with t as a consequence of the properties of the velocity field with position x; the turbulence has the same statistical form, with the corresponding scales $L(x)$ and $V(x)$; t is assumed to be proportional to x^{p+q}. The funda-

mental hypothesis is that the particle velocity can be transformed to a stationary random function, by suitable adjustment of the velocity and time scales $\omega(t)$ and $\tau(t)$; analytically, the hypothesis is that $[u(t)-U_0]/\omega(t)=F(\eta)$, where $F(\eta)$ is a stationary random function of the variable η given by: $d\eta \propto dt/\tau(t)$, U_0 being the translation velocity. From these assumptions the mean position $\bar{x}(t)$ of the particle and $D(t)$, the dispersion around this mean position, are computed; the author gives the values:

jet type flow: $\bar{x}(t)\propto t^{1/(1+\varpi)}$, $D(t)\propto t^{1/(1+\varpi)}$;
wake type flow: $\bar{x}(t)\propto t^{(1-\varpi)/p}$, $D(t)\propto t^{1-q}$.

J. Kampé de Fériet (Lille)

573:

Dubreil-Jacotin, Marie-Louise. Sur le passage des équations de Navier-Stokes aux équations de Reynolds. C. R. Acad. Sci. Paris **244** (1957), 2887–2890.

The paper is devoted to the definition of the average $\bar{f}$ of a function f in order to clarify the connexions between the Navier-Stokes and the Reynolds equations in the theory of turbulence. The author [for her previous work, see the same C. R. **236** (1953), 1136–1138; 1950–1951; **239** (1954), 856–858; MR **14**, 839; **15**, 194; **16**, 145] suggests a replacement of the rules introduced by the reviewer [Ann. Soc. Sci. Bruxelles Sér. I **63** (1949), 165–180; MR **11**, 336] by the following ones: (1) $\overline{f+g}=\bar{f}+\bar{g}$; (2) $\overline{\lambda f}=\lambda\bar{f}$; (3) $\overline{f\bar{g}}+\overline{\bar{f}g}=\bar{f}\bar{g}+\overline{f\bar{g}}$; (4) $\bar{1}=1$; (5) $\partial\bar{f}/\partial\xi=\partial\bar{f}/\partial\xi$. The rule $\bar{f}=\bar{f}$, (equivalent to $f'=0$), which has been usually accepted since Reynolds paper, is not a necessary consequence of the rules (1) to (5).

J. Kampé de Fériet (Lille)

574:

Meecham, W. C.; and Ford, G. W. Acoustic radiation from isotropic turbulence. J. Acoust. Soc. Amer. **30** (1958), 318–322.

Assuming the Mach number of the turbulent motion to be small and following Lighthill's formulation of the problem, the frequency spectrum of the sound radiated by a finite volume of homogeneous isotropic turbulence is derived in terms of the space-time correlation function for the pressure fluctuations,

$$F(\xi,\tau)=\rho_0^{-2}[\langle p(r',t')p(r'',t'')\rangle-\langle p(r',t')\rangle\langle p(r'',t'')\rangle],$$

where ξ and τ are the space and time intervals. As it stands, the behaviour of this function cannot be predicted using the principles of local similarity, but these are shown to be applicable to a "Lagrangian" correlation function in which the two correlation points are considered to move with constant separation at a common velocity equal to the instantaneous average of the velocity over a surrounding volume of order ξ^3. This function is of the form

$$\bar{F}(\xi,\tau)=(\varepsilon\xi)^{4/3}\psi(\varepsilon^{1/3}\xi^{-2/3}\tau)$$

and arguments are presented to show that this function may be substituted in the expression for the sound spectrum without appreciable error. At high frequencies, the spectrum is proportional to $\omega^{-7/3}M^{21/2}$ (ω is the frequency and M the Mach number), and at low frequencies to ω^4M^3. *A. A. Townsend* (Cambridge, England)

575:

***Dubreil-Jacotin, M. L. Sur les axiomes des moyennes.** Corso sulla teoria della turbolenza, Vol. 1, pp. 107–114. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto e Bella, Turin, 1957. viii+339 pp.

On considère un anneau $\mathscr{A}$ de fonctions réelles définies sur un ensemble X et constituant en même temps un espace vectoriel. On suppose que $\mathscr{A}$ contient la fonction $e(x)=1$, $x\in X$. A toute fonction $f\in\mathscr{A}$ on fait correspondre une 'fonction' $\bar{f}\in\mathscr{A}$, dite 'moyenne' de f, et vérifiant les axiomes

$$\text{(L)}\qquad \begin{cases}(1)\ \overline{f+g}=\bar{f}+\bar{g},\\ (2)\ \overline{kg}=k\bar{g},\\ (3)\ \bar{e}=e.\end{cases}$$

On se propose de compléter ces axiomes de telle sorte que, si des fonctions du type f vérifient les équations de Navier-Stokes, leurs moyennes vérifient correctement les "équations moyennes" de Reynolds. Pour cela on ajoute d'abord la condition

$$\text{(D)}\qquad \frac{\overline{\partial f}}{\partial\xi}=\frac{\partial}{\partial\xi}\bar{f}\quad(\xi\text{ composante de }x)$$

puis une dernière condition. Celle qui est d'habitude utilisée par J. Kampé de Fériet s'écrit

$$\text{(T)}\qquad \overline{fg}=\bar{f}\bar{g}.$$

L'auteur se propose de montrer qu'il y a intérêt à remplacer (T) par la condition suivante

$$\text{(R)}\qquad \overline{fg}+\overline{f\bar{g}}=\bar{f}\bar{g}+\overline{\bar{f}g},\ \text{ou}$$

$$\text{(R*)}\qquad \overline{fg}=\bar{f}\bar{g}+\overline{f'g'},\ \text{avec } f=\bar{f}+f',\ g=\bar{g}+g'.$$

D'après des résultats antérieurs, on sait déjà que toute solution de (LTD) vérifie (LRD). On va montrer que sous de larges hypothèses (LTD) n'a pas de solution non constante (autre que $\bar{f}=f$), alors que (LRD) en possède une non triviale.

On suppose que f est une fonction indéfiniment dérivable d'une variable x, définie sur un segment ou sur la droite entière, et vérifiant (LRD). Un calcul simple relatif aux moyennes de x et de x^2 montre que D et R sont compatibles, alors que D et T sont contradictoires. Plus généralement, les conditions (LTD) entraînent que tout polynôme est identique à sa moyenne. Enfin, si l'on adjoint à (LTD) la condition topologique

$$\text{(O)}\qquad f<g\Rightarrow\bar{f}\leq\bar{g},$$

le système d'axiomes (LTDO) n'a pas d'autre solution que $\bar{f}=f$. Au contraire, les axiomes (LRD) admettent la solution

$$\bar{f}=ae^{-ax}\int_{-\infty}^{x}f(t)e^{at}dt,\ a>0.$$

$\bar{f}$ vérifie l'équation différentielle

$$\bar{f}+\frac{1}{a}\frac{d\bar{f}}{dx}=f,$$

qui a déjà été utilisée par les spécialistes de la turbulence à partir d'autres considérations. A l'aide de cette moyenne, le passage des équations de Navier-Stokes aux équations de Reynolds est rigoureusement assuré. *J. Bass* (Paris)

576:

***Arbault, Jean. Transformations de Reynolds sur un ensemble fini.** Corso sulla teoria della turbolenza, Vol. 1, pp. 115–121. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto e Bella, Turin, 1957. viii+339 pp.

On considère un anneau $\mathscr{A}$ de fonctions f réelles définies sur un ensemble X. A $f\in\mathscr{A}$ on associe une fonction $\bar{f}\in\mathscr{A}$ dite moyenne de f et satisfaisant aux régles suivantes,

introduites par J. Kampé de Fériet: 1') $\overline{f+g}=\bar{f}+\bar{g}$, 2') $\overline{\lambda f}=\lambda\bar{f}$, 3') $\overline{fg}=\bar{f}\bar{g}$, 4') $f<g\Rightarrow\bar{f}\leqq\bar{g}$, 5') $\bar{\bar{f}}=\bar{f}$. 5') peut d'ailleurs être remplacé par la condition plus restrictive 5_1') $\bar{e}=e$ où $e=1$ sur X.

Mme Dubreil-Jacotin [voir l'analyse précédante] a montré qu'il y avait intérêt à remplacer 3') par

3) $\overline{f\bar{g}+\bar{f}g}=\bar{f}\bar{g}$, ou $\overline{fg}=\bar{f}\bar{g}+\overline{f'g'}$, avec $f'=f-\bar{f}$, $g'=g-\bar{g}$.

L'auteur se propose de montrer que, si X est un ensemble 'fini' de n éléments, les systèmes d'axiomes A (1, 2, 3) et A' (1', 2', 3', 5') sont équivalents (ce résultat est inexact pour un ensemble infini, par example la droite).

On posera $\bar{f}=Tf$ dans A', $\bar{f}=\mathscr{R}f$ dans A. On peut mettre f sous la forme

$$f=\sum_i \alpha_i C_i.$$

L'expression de $\mathscr{R}f$ résulte alors de celle des $\mathscr{R}C_i$, soit

$$\mathscr{R}C_i=\sum_j \lambda_{ij}C_j.$$

On est ramené à la détermination de la matrice M des λ_{ij}. 3) établit entre les λ_{ij} $\frac{1}{2}n^2(n+1)$ équations, faciles à résoudre pour $n=2$. Dans le cas général, la matrice M a comme valeurs propres 0 et 1. Si 1 est la seule valeur propre, M est l'identité. Dans le cas contraire, quelques théorèmes relatifs à la structure de la matrice M montrent enfin que les axiomes 3') et 5') sont équivalent à l'axiome 3). Mais on doit noter que les axiomes 3) et 3') seuls ne sont pas équivalents entre eux. *J. Bass* (Paris)

577:

Heinz, Carl. Verwundene Überschallflügel und Überschallflügel im Wirbelfeld. Z. Angew. Math. Phys. **9b** (1958), 347–356.

Il s'agit de l'étude d'une aile dans le cadre de l'aérodynamique supersonique linéaire par la méthode de composition de mouvements coniques: cas d'une aile en flèche à profondeur constante dans un écoulement uniforme, cas d'une aile rectangulaire dans un écoulement perturbé par la présence d'un tourbillon rectiligne parallèle à la corde. Dans ce dernier cas on suppose que la réflexion de la perturbation de l'aile sur le tourbillon n'agit pas sur l'aile. Les bords latéraux sont toujours supposés indépendants. *P. Germain* (Paris)

578:

v. Krzywoblocki, M. Z.; Pottsepp, L.; and Saarlas, M. On plane and spatial source in compressible flow. Bull. Tech. Univ. Istanbul **10** (1957), no. 2, 19–27. (Turkish summary)

A discussion is given of the forms of the velocity potential for sources in two and three dimensional compressible flow. The authors maintain that previous discussions given in standard treatises do not propose solutions. Their own discussion, however, is purely formal and would seem to be more an exercise in algebra and integration than a contribution to the mechanics or the mathematics of compressible flow. *Hirsh Cohen* (Delft)

579:

Tyler, R. D. One-dimensional treatment of non-uniform flow. Aero. Res. Council, Rep. and Memo. no. **2991** (1954), 20 pp. (1957).

The usual one-dimensional treatment of perfect gas flows assumes a uniform velocity profile and thus a uniform distribution of the other flow parameters across the duct. This notion is tested by introducing the type of non-uniform distributions which are caused by the presence of the boundary layer. Mean flow quantities are defined and, using them, a comparison is made of the uniform and non-uniform flows in three typical design and analysis cases. In general, the results show that the errors involved in using the usual uniform flow theory are small for reasonable boundary layer thicknesses and into the supersonic range. Exceptions to this finding are cited. *Hirsh Cohen* (Delft)

580:

Kanwal, R. P. Variation of flow quantities along streamlines and their principal normals and binormals in three-dimensional gas flows. J. Math. Mech. **6** (1957), 621–628.

A formal discussion is given of three dimensional steady rotational compressible gas flows. Relations between the geometrical properties of the streamline trajectories and the flow quantities are set out. One result is that the pressure along the binormals to the streamline is constant. A three dimensional analog of Croccos' formula connecting vorticity, streamline curvature and entropy change is found. *Hirsh Cohen* (Delft)

581:

Daskin, Walter; and Feldman, Lewis. The characteristics of two-dimensional sails in hypersonic flow. J. Aero. Sci. **25** (1958), 53–55.

A sail is a structure which carries its load primarily in tension and not, as a conventional wing, in bending. This paper gives a theory for a two-dimensional sail in hypersonic flow. The latter condition allows a simple expression to be given for the pressures on the upper and lower surfaces of the sail. Using this the force coefficients for the class of sails considered is computed. It is pointed out that lift and drag coefficients for sails can be made nearly equal to those of conventional surfaces at hypersonic speeds but that the sails may have weights from 1/5 to 1/10 those of the conventional lifting surfaces. *Hirsh Cohen* (Delft)

582:

Vaglio-Laurin, Roberto; and Van Dyke, M. D. A discussion of higher-order approximations for the flow field about a slender elliptic cone. J. Fluid Mech. **3** (1958), 638–644.

The method of linearized characteristics has appeared to be useful for determining the pressure distribution on non-axially symmetric bodies, for example. It has been criticised by Van Dyke [same J. **1** (1956), 1–15; MR **18**, 846] on the basis of a comparison of the results obtained for an elliptic cone of 3:1 axis ratio at $M=\sqrt{2}$ by second order slender-body theory and by applying to this theory the approximation procedure inherent in the linearized characteristics method (linearizing with respect to deviation of the cross-section from circular). In the first part of the present paper, Vaglio-Laurin shows that provided an area-rule requirement is satisfied, reasonably good results can be obtained from linearized characteristics even for the above relatively unfavourable case. Van Dyke concedes this in the second half of paper but maintains the superiority of second-order slender-body theory (supported by some experimental results). *H. C. Levey* (Nedlands)

583:

***Gupta, A. S. A note on laminar motion due to the oscillation of a flat plate in a compressible fluid.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 195–198. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

This is the extension of the classical problem of the periodic motion of a viscous fluid bounded by an oscillating plane wall to the case of a compressible fluid with heat transfer. Prandtl number is assumed constant and viscosity proportional to the temperature. The method used is to introduce a stream function ψ and to write the energy and momentum equations with ψ and the time t as independent variables. The results show that the heat transfer is not affected by the oscillation and the skin friction is not affected by compressibility. The case of a fluid bounded by a rigid plane wall is also treated.

W. R. Sears (Ithaca, N.Y.)

584:

Bruhn, Gerhard; und Haack, Wolfgang. Ein Charakteristikenverfahren für dreidimensionale instationäre Gasströmungen. Z. Angew. Math. Phys. **9b** (1958), 173–190.

The characteristic equations for the unsteady three-dimensional motion of inviscid perfect gas are derived. Some results are shown of a computation of the early stages of a two-dimensional expansion wave travelling into gas at rest in a nozzle. The motion is generated by steady suction at the nozzle exit, in one case, and by bursting of a diaphragm there with large pressure ratio, in the other case, for which violent lateral pressure waves are indicated.

R. E. Meyer (Providence, R.I.)

585:

Roseau, Maurice. Sur les solutions d'un problème aux limites de type mixte. C. R. Acad. Sci. Paris **246** (1958), 226–227, 369–371.

L'auteur donne des formules explicites pour les solutions du problème suivant:

$\Delta\varphi-k^2\varphi=0$ dans le secteur xOt (la demidroite Ot définie par $(Ox\ Ot)=-\alpha$, $0<\alpha<\frac{1}{2}\pi$, k réel positif);

$\partial\varphi/\partial y-\varphi=0$ sur Ox, $\sin\alpha(\partial\varphi/\partial x)+\cos\alpha(\partial\varphi/\partial y)=0$ sur Ot;

φ bornée à l'infini dans le secteur, et régulière dans le secteur, sauf à l'origine où elle a une singularité.

Pour $k=0$ (première Note) ces formules sont de la forme:

$$\varphi_j^{(p)}=\mathrm{Re}\,\lambda_j\int_{C_j}e^{\zeta z}\,\frac{g(\zeta)}{\zeta+i}\,d\zeta,\quad z=x+iy,\quad r=|z|,$$

$j=1, 2$, $\lambda_1=1$, $\lambda_2=i$. Pour les définitions de C_j et $g(\zeta)$ voir Roseau, même C. R. **232** (1951), 211–213 [MR **12**, 869].

Pour $k<1$ les formules sont:

$$\varphi_j^{(p)}=\mathrm{Re}\,\lambda_j\int_{C_j}\exp\frac{k}{2}\left\{x\left(\zeta+\frac{1}{\zeta}\right)+iy\left(\zeta-\frac{1}{\zeta}\right)\right\}\times\zeta^{p(\pi/\alpha)}\,\frac{g(\zeta)d\zeta}{\frac{1}{2}k(\zeta-\zeta^{-1})+i};$$

λ_j, C_j et $g(\zeta)$ définis dans Roseau, loc. cit.

Une des solutions $\varphi_1^{(p)}$, $\varphi_2^{(p)}$ au moins est singulière d'ordre $r^{-p(\pi/\alpha)}$ à l'origine. L'application au problème des ondes liquides sur une plage montre que prescrivant l'onde de surface à l'infini et une singularité à l'origine d'ordre $r^{-q(\pi/\alpha)}$ il existe une infinité de solutions qui dépendent linéairement de q constantes arbitraires.

Pour $k>1$ l'A. construit des solutions par une formule analogue à (2) (pour la définition de $g(\zeta)$ voir alors Roseau ibid. **245** (1957), 2472–2474 [MR **19**, 1218]) ces solutions tendent vers σ à l'infini. Ce comportement à l'origine est de même nature que dans le cas $k<1$.

Y. Fourès-Bruhat (Marseille)

586:

Litwiniszyn, Jerzy. Flows with the exchange of mass, momentum and energy. Arch. Mech. Stos. **9** (1957), 669–683. (Polish and Russian summaries)

Equations for one-dimensional flow with addition or subtraction of mass, momentum, and energy along the stream tube are written down, first in Lagrangian and then in Eulerian coordinates. Examples treated are (1) incompressible flow with mass addition and (2) perfect-gas flow, originally steady and uniform, but with introduction of mass and energy along the tube beginning at a certain instant.

W. R. Sears (Ithaca, N.Y.)

587:

Zav'yalov, Yu. S. On a certain class of plane, steady, rotational gas flows. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 363–364. (Russian)

The author shows that if the velocity $v(p, \psi)$ (p pressure, ψ stream function) is expressed in the form

$$(*)\qquad v(p, \psi)=f_1(p)F_1(\psi)+f_2(p)F_2(\psi),$$

where f_1 and f_2 satisfy the condition $f_1''(p)/f_1(p)=f_2''(p)/f_2(p)$, then the rotational motion of a gas is reducible to a potential motion. If we take any function $V(p)$ such that $V''(p)/V(p)=v^{-1}(\partial^2 v/\partial p^2)$ and define a new variable ψ^* by $V(p)(\psi^*-\psi_0^*)=\int_{\psi_0}^{\psi}v(p, \psi)d\psi$, then the equation of Sedov and Rudnev becomes

$$V(p)\frac{\partial^2\psi^*}{\partial p^2}-V''(p)\frac{\partial^2\psi^*}{\partial\theta^2}+2V'(p)\frac{\partial\psi^*}{\partial p}=0,$$

where θ is the angle of inclination of the velocity vector to the x-axis. The analysis reduces to cases previously studied when (*) is $v(p, \psi)=V(p)F(\psi)$.

C. D. Calsoyas (Livermore)

588:

Nikol'skiĭ, A. A. The action of the force due to the "second" form of hydrodynamic motion on flat bodies (the dynamics of plane discontinuous flows). Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 365–368. (Russian)

589:

Nonweiler, T. R. F. The sonic flow about some symmetric half-bodies. J. Fluid Mech. **4** (1958), 140–148.

On the hodograph method, a compressible fluid flow which is slightly disturbed from a uniform stream at sonic velocity a^* is specified approximately by the equations $\Phi_u=u\Psi_v$, $\Phi_v=-\Psi_u$, where u, v are the cartesian velocity perturbations and Φ, Ψ the velocity potential and stream function. In the present paper the variables $w=(4/9)u^3+v^2$, $\xi=v^2/w$ are used, and the flow represented by

$$\Phi=-a^*l(\tfrac{2}{3})^{1/3}w^{-m+2/3}(1-\xi)^{2/3}F(m, \tfrac{7}{6}-m; \tfrac{1}{2}; \xi)$$

$$\Psi=2a^*l(m-\tfrac{2}{3})w^{-m+1}\xi^{\frac{1}{2}}F(m, \tfrac{4}{6}-m; \tfrac{3}{2}; \xi)$$

is discussed, l, m being constants. If $m\geqq 3/2$ this gives a flow past a profile on which $\pm yx^{-n}$ is constant, where $n=(6m-7)/(6m-4)$, $\frac{2}{3}\leqq n<1$. If $n=2/5$ the speed along this profile is constant and, in fact, sonic. The pressure along the profile is deduced, and the singularity at the nose (where the initial approximation breaks down) is discussed. In particular, when $n=2/5$ the existence of an 'edge force' at the nose is deduced.

T. M. Cherry (Melbourne)

590:

***Carrière, P. Méthodes théoriques d'étude des écoulements supersoniques. Préface de G. Darrieus.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 339, 1957. xi+234 pp. 3000 francs.

This is a very clear and detailed exposition of the numerical method of characteristics for calculating steady plane and axisymmetric supersonic flows. Full treatment is given to such related subjects as one-dimensional flow relations, shock waves, the circular cone, the flow near a sonic throat, and the relations between the curvatures of a shock wave and the streamlines traversing it, some of these last results being new. Extensive tables and charts are included. *M. D. Van Dyke* (Paris)

591:

Lock, R. C. An extension of the linearized theory of supersonic flow past quasi-cylindrical bodies, with applications to wing-body interference. J. Fluid Mech. **4** (1958), 33–63.

This is an extension of the methods of Nielsen [NACA Rep. 1252 (1955)], Randall [Royal Aircraft Estab. Tech. note no. Aero 2404, and British Aero. Res. Coun. Tech. note no. 18492 (1955)], and others concerning the supersonic flow fields of nearly cylindrical slender bodies. The perturbation field is expressed with the aid of the Heaviside calculus and the boundary condition at the body is expanded in a Fourier series. Although the resulting solution, as found by Nielsen, is not very satisfactory for computation of the pressure field as a whole, it is practical for computation of the pressures on the body, with the aid of functions tabulated by Randall. Here it is shown that certain integrals of the pressure along lines parallel to the y axis in the xy plane can be expressed in terms of Randall's functions. (Rectangular Cartesian axes xyz are used, the x axis lying along the body axis.) If a wing of suitable geometry (supersonic leading edge, trailing edge parallel to the y axis, sufficiently large aspect ratio, etc.) is mounted on the body, these integrals permit evaluation of the effects of body geometry on wing forces and moments. A problem of wing-body interference (studied by Nielsen) is also simplified by this method. Nielsen's results are evaluated by comparison and are found to contain errors for large values of the ratio of wing chord to body radius. A comparison with the supersonic area rule proposed by R. T. Jones [NACA Rep. 1284 (1953)] is also afforded. It is concluded that the interference correction given by the area rule may actually be of the wrong sign. The limitations of the area rule were previously discussed by Lomax and Heaslet [J. Aero. Sci. **23** (1956), 1061–1074; MR **18**, 620]. *W. R. Sears* (Ithaca, N.Y.)

592:

Van Dyke, Milton D. The supersonic blunt-body problem — review and extension. J. Aero. Sci. **25** (1958), 485–496.

The review part of this paper considers (i) potential flow approximations, (ii) series expansions in distance from an assumed shock, (iii) incompressible (but not irrotational) approximations, (iv) Newtonian approximation and its improvements. Briefly, the conclusions are that (i) gives only a very mediocre approximation which is much improved when rotational effects can be included as in (iii), method (ii) cannot be used in many cases (without some form of analytic continuation) because the "limit line" upstream of the shock is nearer than the body, and in the case of (iv) the convergence of systematic expansions in powers of $\gamma-1$ is disappointingly slow.

The remainder of the paper is devoted to numerical methods of calculating solutions and, in particular, to a presentation of the author's successful work in this field. His method uses a finite difference scheme to calculate the flow (and thus the body shape) behind shocks with shapes given by conic sections. Simple forward differences are used in the direction normal to the shock but eleven point differences are used parallel to the shock. This indirect problem is "improperly posed" in the subsonic region and stability troubles can arise. However, in this event, it turns out that the growth of errors is not sufficient to swamp the answers, provided not too many steps are used between shock and body. In a typical case an initial error is found to be multiplied by 160 in the 11 steps used, with a maximum of about 3 per step near the stagnation point. *G. B. Whitham* (New York, N.Y.)

593:

Whitham, G. B. On the propagation of shock waves through regions of non-uniform area or flow. J. Fluid Mech. **4** (1958), 337–360.

The author presents a rule whereby the propagation of shock waves can be completed easily and accurately when disturbances overtaking the shock have negligible effect on its behaviour. The rule is that the characteristic family more nearly parallel to the shock is applied along the shock wave. This, together with the shock conditions, provides sufficient equations to determine the shock behaviour.

The rule is verified by comparison with solutions, obtained by more laborious techniques, of a wide range of problems. From a study of these solutions the author seeks an explanation of the efficacy of his rule, but, while clarifying certain points, is unable to find the underlying reason.

The paper is frankly speculative but the reviewer would like to question one aspect. In the application to shock expansion theory, the basic assumption regarding overtaking disturbances is not satisfied. However, an accurate solution is obtained by applying the rule not, as seems implied by the author, to the shock wave but to the particle path. In fact, the application of the rule to the shock wave gives an inaccurate description of its propagation. *J. J. Mahony* (Sydney)

594:

Metelicyn, I. I. Principle of least constraint in shock theory. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 11, 105–115. (Russian)

A principle of least constraint, $\delta Z'=0$, applicable to the motion of mechanical systems under impulsive forces is derived. The "constraint" here has the form

$$(*) \qquad Z'=\tfrac{1}{2}[\sum_s m_s(\mathbf{v}_s-\mathbf{v}_s^0)^2-\sum_s \mathbf{P}_s(\mathbf{v}_s-\mathbf{v}_s^0)],$$

where m_s is the mass of a typical particle in the system, $\mathbf{v}_s$ is its actual velocity, $\mathbf{v}_s^0$ its initial velocity, and $\mathbf{P}_s$ the impulse acting on it. With different assumptions about the physical properties of the bodies involved one can deduce from (*) known results from the theory of shock. The author treats a succession of continuous body cases – shocks applied to a rigid body, to a perfect fluid, to a rigid body floating in a perfect fluid, to one end of an elastic rod, and (in considerable detail) to the edge of a thin plate. For each case an appropriate form of Z' is found, and examples of the use of the principle of least constraint are given for some.

R. N. Goss (San Diego, Calif.)

595:

Jarre, Gianni. The dissociation of a pure diatomic gas behind a strong normal shock wave. Z. Angew. Math. Phys. **9b** (1958), 389–403.

L'auteur étudie la dissociation d'un gaz diatomique sous l'effet d'un choc normal violent. Le phénomène est une réaction endothermique et la dissociation est importante seulement si l'élévation de température compense la diminution de pression. Les phénomènes de transfert et d'ionisation étant ignorés, l'auteur établit que les discontinuités à travers le choc dépendent seulement du nombre de Mach amont tandis que l'évolution du fluide après le choc est fonction de l'effet thermique et de l'effet de compressibilité. *H. Cabannes* (Marseille)

596:

Shapiro, Ascher H. A note on the vorticity downstream of a curved shock. Z. Angew. Math. Phys. **9b** (1958), 637–641.

L'auteur considère le mouvement plan permanent d'un fluide compressible non visqueux, comportant une onde de choc. Le mouvement avant le choc étant uniforme (de vitesse V), l'auteur étudie la variation du tourbillon ω après le choc; R désignant le rayon de courbure du choc, la quantité $\omega R/V$ est calculée en fonction du nombre de Mach amont et de la direction du choc.

H. Cabannes (Marseille)

597:

Kanwal, R. P. Existence and uniqueness of flows behind three-dimensional stationary and pseudo-stationary shocks. Proc. Amer. Math. Soc. **9** (1958), 201–207.

Les équations régissant les écoulements stationnaires en aval d'une onde de choc sont écrites dans un système de coordonnées pour lequel les lignes de courant sont des lignes coordonnées, et le choc une surface coordonnée. L'auteur constate, comme il est connu, que les conditions de choc définissent des données de Cauchy pour le système d'équations ainsi obtenu. Ce résultat est généralisé aux écoulements pseudo-stationnaires. *P. Germain* (Paris)

598:

Craggs, J. W. The supersonic motion of an aerofoil through a temperature front. J. Fluid Mech. **3** (1957), 176–184.

The solution obtained in the author's paper reviewed below is applied to the case where the two fluids differ only in the temperatures $T_{1,2}$, and the resulting s_2 is given as a function of $\mathrm{Re}(\zeta_2)$ to first order in $(T_2-T_1)/T_1$. By superposition, the author obtains then the distribution of s along the surface of a symmetrical airfoil passing through the temperature front and, hence, the transient change in drag. Numerical results for a double-wedge airfoil are tabulated. *G. Kuerti* (Cleveland, Ohio)

599:

Craggs, J. W. The oblique reflexion of sound pulses. Proc. Roy. Soc. London. Ser. A. **237** (1956), 372–382.

A solution of the two-dimensional wave equation, representing the reflection and diffraction of a weak shock-wave (pulse) which is produced by the supersonic motion (velocity V) of a thin wedge moving in the y-direction across the boundary $y=0$ of two fluids in $y\lesseqgtr 0$ with different physical constants $c_{1,2}$ (sound velocity) and $\rho_{1,2}$ (density). The boundary-value problem is formulated for the unknown condensations $s_{1,2}=\Delta\rho_{1,2}/\rho_{1,2}$ in polar coordinates θ and $\lambda=r/t$ (conical field), the equations being hyperbolic outside the semicircles $(\lambda=c_2,\ 0<\theta<\pi)$, $(\lambda=c_1,\ 0>\theta>-\pi)$, and elliptic inside; and readily transformed into standard forms. The physical situation that would lead to total reflection in the case of an oblique infinite wave train corresponds here to the assumption $c_1<c_2\cos\varphi_1$, φ_1=Mach angle $\arcsin c_1/V$. In that case, the "immovable" image of the primary pulse in the lower half-plane intersects the boundary $\theta=0$ between $\lambda=c_1$ and $\lambda=c_2$ (points A and B) at $\lambda=c_1/\cos\varphi_1$ (point H).

The solution in the hyperbolic regions is easily determined except in the area between the arc $\lambda=c_1$ and the characteristic through B, the range of influence of the segment AB. In the elliptic regions, $s_{1,2}$ are considered as real parts of analytic functions $w_{1,2}$. Continuity of pressure and normal velocity along AB leads to a relation (1) between the λ-derivatives of $\mathrm{Re}(w_2)$ and $\mathrm{Im}\,(w_2)$ along AB, and, after mapping of the semi-circles onto half-planes $(\zeta_{1,2})$, an intricate discussion of the possible singularities of $dw_2/d\zeta_2$, based on (1), permits the complete determination of $dw_2/d\zeta_2$; $dw_1/d\zeta_1$ is then found from the boundary conditions along the common real axis.

The principal result is that the total reflection of a pulse establishes a diffuse wave which precedes the primary pulse, corresponding in the λ, θ-plane to the range of influence of the segment HB.

G. Kuerti (Cleveland, Ohio)

600:

Olsen, Haakon; Wergeland, Harald; and Westervelt, Peter J. Acoustic radiation force. J. Acoust. Soc. Amer. **30** (1958), 633–634.

A formula for the force is derived without making any assumptions about the scatterer. *Y. H. Kuo* (Peking)

601:

Voĭt, S. S. Propagation of sound in a medium moving with a velocity of varying direction. Trudy Morsk. Gidrofiz. Inst. **10** (1957), 10–16. (Russian)

A horizontal layer of fluid of thickness h separates two semi-infinite bodies of the fluid flowing horizontally at equal steady speeds U, but in different directions. The flow direction in the intermediate layer changes linearly with vertical height from that in the lower body to that in the upper. A plane sound wave originating in the lower body is incident on the intermediate layer. Equations of equilibrium and continuity are set up, neglecting higher order terms. Conditions of continuity of pressure and vertical particle velocity at the interfaces are satisfied and a periodic solution obtained for amplitudes of transmitted and reflected waves, with the help of a power series in terms of the small parameter U/c (c the velocity of sound). *W. W. Soroka* (Berkeley, Calif.)

602:

Maidanik, G. Torques due to acoustical radiation pressure. J. Acoust. Soc. Amer. **30** (1958), 620–623.

A general expression for the torque exerted by acoustical radiation pressure is calculated by a method similar to that of Westervelt [J. Acoust. Soc. Amer. **23** (1951), 312–315; MR **12**, 875]. *Y. H. Kuo* (Peking)

603:

Lyamšev, L. M. On the scattering of sound by a thin rod of finite length. Akust. Ž. **4** (1958), 51–58. (Russian)

By a rod "of finite length" is meant the portion of an infinite cylinder contained between two rigid and immovable plane sections. In the field of a plane sound wave at an arbitrary angle of incidence the rod gives rise to a

scattered field comprising a component due to an absolutely rigid infinite rod and a component induced by the vibration of the finite rod. The latter is reduced to the sum of two integrals $X(r, \phi, z)$ over the displacement velocities associated with the flexural and extensional modes of vibration, respectively. The total pressure field is then required to satisfy the equations of motion of the two modes. Solution of the resulting system for the velocities is obtained in the form of expansion in eigenfunctions of the rod vibrating in vacuum. A study of the integrals X reveals the theoretical existence of strong scattering in the direction opposite to θ, the angle of incidence, for certain values of θ.

R. N. Goss (San Diego, Calif.)

604:

Lysanov, Yu. P. On the scattering of sound on inhomogeneous surfaces. Akust. Ž. **4** (1958), 47–50. (Russian)

A limitation in the author's earlier work [same Ž. **1** (1955), 58–69; MR **17**, 684], namely, that the solution there obtained is valid only for a surface whose acoustic conductance departs but little from its mean value, is removed for the case $W=\cos qx$ under normal incidence of a plane wave. The infinite system of equations in the complex amplitudes of the scattered waves is solved by a method of successive approximations, those for the specularly reflected wave being carried out in detail. A solution of prescribed accuracy for the case cited is obtained by considering a reduced system, the finite order of which depends upon the surface parameters and the wavelength. *R. N. Goss* (San Diego, Calif.)

605:

Oldroyd, J. G. Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. Proc. Roy. Soc. London. Ser. A. **245** (1958), 278–297.

An invariant constitutive equation describing the general motion of an isotropic, incompressible visco-elastic liquid displaying significant non-Newtonion effects, such as the Weissenberg, Poynting and Kelvin effects, is proposed as a generalization of Oldroyd's well known linear three-constant equation containing only one viscosity coefficient as well as a stress-relaxation-time and a strain-retardation-time. The equation contains 8 physical constants of dimension of time and includes terms of second order in the combined stresses and velocity gradients, as well as in the velocity gradients alone, with material derivatives of the stress-deviation and the rate-of-strain deviation, taking into account the linear and angular motion of the fluid element.

The consequences of this equation are investigated with respect to the Weissenberg and Poynting effects in steady flow between vertical coaxial cylinders, in rectilinear flow and in shear between two coaxial cones as well as between a cone and a flat plate.

It is shown that with two of the eight constants vanishing the remaining six parameter equation represents the main non-Newtonian features of several observed visco-elastic solutions.

A. M. Freudenthal (New York, N.Y.)

606:

Slibar, A.; und Paslay, P. R. Die axiale Strömung von Bingham Plastiken in konzentrischen Rohren. Z. Angew. Math. Mech. **37** (1957), 441–449. (English, French and Russian summaries)

The Buckingham-Reiner solution of laminar axial visco-plastic flow in a cylindrical tube is extended to such flow between two concentric tubes using the 3-dimensional v. Mises yield condition. Flow occurs in two concentric rings adjacent to the external and internal tube walls. A concentric rigid core moves with constant axial velocity between these regions.

A. M. Freudenthal (New York, N.Y.)

607:

Lüst, R.; und Schlüter, A. Axialsymmetrische magnetohydrodynamische Gleichgewichtskonfigurationen. Z. Naturf. **12a** (1957), 850–854.

The condition for hydromagnetic equilibrium in the axially symmetric case leads to a second-order differential equation for the poloidal component of the field [Chandrasekhar and Prendergast, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 5–9; MR **17**, 561]. This equation becomes linear under special assumptions about the pressure and the toroidal component of the field. The case of a constant toroidal field and a pressure field that is a quadratic function of the poloidal field is investigated in detail in this paper. The differential equation is solved explicitly and the configuration of the magnetic lines of force is calculated numerically and exhibited in a figure in a special case. Finally, the stability criterion of Hain, Lüst and Schlüter [see paper reviewed above] is used to investigate the stability of the configuration.

D. Layzer (Cambridge, Mass.)

608:

Hain, K.; Lüst, R.; und Schlüter, A. Zur Stabilität eines Plasmas. Z. Naturf. **12a** (1957), 833–841.

This paper presents an exceedingly general and thorough discussion of the stability against small disturbances of hydromagnetic equilibrium in an inviscid, perfectly conducting fluid. The stability is shown to depend on the sign of a Hermitian quadratic form in the velocities. When the stress tensor is isotropic, the rate of growth of an instability is shown to have a definite upper bound, which is given explicitly. In some special cases the stability criterion becomes quite simple: for instance, for meridional fields with cylindrical symmetry. The stability problem for a cylinder inside of which the field vanishes is solved explicitly. Finally, the stability criterion is generalized to the case of an anisotropic stress tensor.

D. Layzer (Cambridge, Mass.)

609:

Trehan, S. K. The hydromagnetic oscillations of twisted magnetic fields. I. Astrophys. J. **127** (1958), 446–453.

Chandrasekhar [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 273–276; MR **18**, 357] has shown that the stationary solution of the equations governing a perfect fluid conductor in a magnetic field, with $V=H/(4\pi\rho)^{\frac{1}{2}}$, is always stable against small perturbations. On the other hand, it is well known that static solutions of the equations may be either stable or unstable. This paper investigates the case of a helical field in the interior of an infinitely long cylinder, where the stability was shown by Roberts [Astrophys. J. **124** (1956), 430–442] to depend in the static case on the ratio between the pitch of the magnetic lines of force and the radius of the cylinder, with a view to showing explicitly how the fluid motions lead to stability. It is shown that in the stationary case the normal component of the gradient of the perturbation pressure, at the boundary of the fluid, always opposes the normal component of the fluid displacement, while in the static case this is not so. *D. Layzer* (Cambridge, Mass.)

610:
Trehan, S. K.; and Reid, W. H. The hydromagnetic oscillations of twisted magnetic fields. II. Astrophys. J. 127 (1958), 454–458.

This paper treats a generalization of the problem treated in the paper reviewed above. The fluid and the magnetic field extend to infinity instead of being confined, as in the preceding case, to the interior of a cylinder. The generalization is found not to affect the periods of oscillation markedly. The stability is explicitly exhibited, as in the preceding paper.

D. Layzer (Cambridge, Mass.)

611:
Taniuti, Tosiya. On the propagation of the hydromagnetic waves in compressible ionized fluid. Progr. Theoret. Phys. **19** (1958), 69–76.

This paper discusses the propagation of finite-amplitude disturbances in an infinitely conducting medium pervaded by a uniform magnetic field. The discussion is based on the method of characteristics and proceeds along lines that are familiar in the theory of hydrodynamic shock waves and supersonic flow [R. Courant and K. O. Friedrichs, Supersonic flow and shock waves, Interscience, New York, (1948); MR **10**, 637]. Particular attention is given to the development of surfaces of discontinuity.

D. Layzer (Cambridge, Mass.)

612:
Nardini, Renato. Sulla mutua azione fra fenomeni acustici ed idromagnetici. J. Math. Mech. **7** (1958), 1–15.

This paper treats, in the usual way, the propagation of small-amplitude disturbances in a perfect fluid of infinite conductivity pervaded by a magnetic field. It differs from previous discussions in that the field is taken to be that of a straight wire carrying a uniform and constant current, instead of the usual spatially constant field.

D. Layzer (Cambridge, Mass.)

613:
Chandrasekhar, S. The partition of energy in hydromagnetic turbulence. Ann. Physics **2** (1957), 615–626.

This paper treats a special point in the author's theory of hydromagnetic turbulence [Proc. Roy. Soc. London Ser. A **233** (1955), 322–330; MR **17**, 920]: the evaluation of the ratio between the amplitudes of the magnetic- and kinetic-energy spectra in the inertial subrange. (On the author's theory both spectra have the same form.) The author explains how the boundary conditions on the fundamental differential equations lead to a characteristic-value problem for this ratio. The differential equations have two movable singularities and in order for the boundary conditions to be satisfied, the two singularities must coincide. This condition determines the ratio mentioned above, which turns out to have the value 1.6265.

D. Layzer (Cambridge, Mass.)

614:
Zumino, Bruno. Some questions in relativistic hydromagnetics. Phys. Rev. (2) **108** (1957), 1116–1121.

The author formulates the basic differential equations of hydromagnetics in a Lorentz invariant form and discusses some of their general properties. He then passes to the associated integral equations, from which he derives the shock conditions (relations between the values of the dependent variables on either side of a surface of discontinuity), also in Lorentz invariant form. Finally he obtains an explicit solution to the problem of weak hydromagnetic shocks. Though many of the results have been obtained previously (the paper contains references to previous work), the elegant, completely relativistic account contained in the present paper serves to clarify the mathematical foundations of the subject.

D. Layzer (Cambridge, Mass.)

615:
Stewartson, Keith. Toroidal oscillations of a spherical mass of viscous conducting fluid in a uniform magnetic field. Z. Angew. Math. Phys. **8** (1957), 290–297.

The problem named in the title of this paper had been previously considered by Plumpton & Ferraro [Astrophys. J. **121** (1955), 168–174; MR **17**, 199] for the case of zero viscosity and infinite conductivity. They found the intuitively obvious result that every shell formed by rotating a line of force about the axis of symmetry oscillates with its own characteristic frequency. In general, this characteristic frequency does not have a constant value throughout the sphere but varies continuously from one shell to the next. The author of the present paper points out that the solution of Plumpton and Ferraro leads to certain formal contradictions, the most important of which expresses the evident fact that the governing differential equations cannot have as a regular solution a field that vanishes everywhere except on a given surface. To resolve this difficulty, the author treats the eigenvalue problem for a viscid fluid of finite conductivity and obtains approximate expressions for the oscillating fields and the characteristic periods. The oscillations turn out to have appreciable amplitudes only near the axis of symmetry, and as the conductivity is increased the region of non-negligible amplitudes shrinks, until for a perfect conductor it coincides with the axis. In other words, the only rigorous solution to the problem treated by Plumpton and Ferraro is one in which both the velocity and magnetic fields vanish identically, i.e., no oscillation at all.

D. Layzer (Cambridge, Mass.)

616:
Resler, Edwin L., Jr.; and Sears, William R. Magnetogasdynamic channel flow. Z. Angew. Math. Phys. **9b** (1958), 509–518.

Dans un article déjà analysé [J. Aero. Sci. **25** (1958), 235–245, 258; MR **19** 1226] les auteurs obtenaient pour l'écoulement d'un fluide conducteur dans une tuyère à l'approximation quasi-rectiligne, des expressions de l'accélération et de la variation du nombre de Mach, moyennant les hypothèses simplificatrices usuelles.

Le présent article est une étude détaillée des deux cas suivants: (1) la section de la tuyère varie de façon à ce que la densité demeure constante. Une solution exacte est obtenue. (2) La section est constante. Les paramètres étant les valeurs de la vitesse et du nombre de Mach à l'entrée, et le rapport E/H des champs électriques et magnétiques appliqués, les écoulements ne présentant pas de point d'arrêt se répartissent en deux classes: (a) Écoulements non transoniques, accélérés ou décélérés. La vitesse tend alors asymptotiquement vers le rapport E/H. (b) Écoulements transoniques: ils peuvent théoriquement être accélérés ou décélérés, réalisant ainsi magnétiquement un écoulement de Laval.

J. Naze (Marseille)

617:
***Chopra, K. P.; and Singer, S. F. Drag of a sphere moving in a conducting fluid in the presence of a magnetic field.** 1958 Heat transfer and fluid mechanics institute, held at University of California, Berkeley, Calif., June, 1958: preprints of papers, pp. 166–175. Stanford University Press, Stanford, Calif., 1958. viii+264 pp. $7.50.

This paper reports without proof an extension of the

results of an earlier paper [Indian J. Phys. **30** (1956), 605–610; MR **18**, 699]. Here the direction of translation and the axis of rotation need not be parallel to the magnetic axis of the sphere.

A. A. Blank (New York, N.Y.)

618:

Biermann, L.; Hain, K.; Jörgens, K.; und Lüst, R. Axialsymmetrische Lösungen der magnetohydrostatischen Gleichung mit Oberflächenströmen. Z. Naturf. **12a** (1957), 826–832.

The problem solved is that of determining a harmonic vector B (div $B=0$, curl $B=0$) in the domain exterior to a circular torus, given the boundary condition on the normal component, $B_n=0$, and fixing the period $I=\oint_\Gamma B\cdot ds$, where Γ is any simple closed curve which links the torus. An approximate solution is obtained for the case in which the diameter of the cross-section is small compared with distance from the axis of rotational symmetry. An expansion is obtained for the exact solution. *A. A. Blank* (New York, N.Y.)

619:

Zeuli, Tino. Sul vortice magneto idrodinamico cilindrico circolare. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957–58), 105–114.

Etant donné un fluide incompressible doué de conductivité électrique, l'auteur construit une solution des équations de l'écoulement qui représente un tourbillon cylindrique circulaire. Ce résultat est inspiré de la théorie du tourbillon d'Helmholtz en hydrodynamique classique.

H. Cabannes (Marseille)

620:

***Agostinelli, Cataldo. Turbolenza in magneto idrodinamica.** Corso sulla teoria della turbolenza, Vol. 1, pp. 289–335. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto e Bella, Turin, 1957. viii+339 pp.

Ce mémoire est un exposé général sur la turbulence en magnéto-hydrodynamique. Il est divisé en trois chapitres: 1. équations de la magnéto-hydrodynamique (équations générales, équations de Navier-Stokes, tenseur des efforts électromagnétiques, analogie entre le champ magnétique et les tourbillons, énergie électromagnétique dans un mouvement turbulent); 2. théorie de la turbulence isotrope (équations fondamentales, corrélation entre la vitesse et le champ magnétique, équations en fonction du potentiel vecteur); 3. dissipation de l'énergie et fluctuations de la pression (dissipation de l'énergie par viscosité et conductivité, corrélation entre la pression et le champ magnétique). *H. Cabannes* (Marseille)

621:

***Lykoudis, Paul S. On a class of magnetic laminar boundary layers.** 1958 Heat transfer and fluid mechanics institute, held at University of California, Berkeley, Calif., June, 1958: preprints of papers, pp. 176–186. Stanford University Press, Stanford, Calif., 1958. viii+264 pp. $7.50.

Aux vitesses supersoniques, derrière une onde de choc détachée, l'énergie cinétique est transformée essentiellement en énergie thermique et chimique; il en résulte une dissociation et une ionisation du gaz. En se plaçant dans ces conditions, l'auteur étudie l'écoulement autour d'un dièdre; il prouve l'existence pour les équations de la couche limite de solutions homogènes, dans le cas où la composante du champ magnétique perpendiculaire à l'écoulement varie comme une puissance de la distance au point de stagnation. *H. Cabannes* (Marseille)

622:

Pikel'ner, S. B. Fundamental concepts of magneto-hydrodynamics. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958**, 46–53. (Russian)

A brief account is given of the concepts, principally the influence of the motion of the medium on the magnetic field and the effect of the field on the medium, underlying magneto-hydrodynamics. (Author's summary.)

K. C. Westfold (Pasadena, Calif.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 135, 420, 516, 607, 608, 618, 728.

623:

Chako, Nicholas. Characteristic curves on planes in the image space. Chalmers Tekn. Högsk. Handl. no. **191** (1957), 50 pp.

The purely geometrical performance of symmetrical optical systems may be described in terms of curves which represent the points of intersection with a chosen image plane (or surface) of a family of rays whose members issue from a given object point and pass through a circle lying in the plane of the exit pupil and concentric with the axis of the system. Such curves are called characteristic curves in the present paper. [They have previously been investigated fairly fully, for instance, by G. C. Steward, The symmetrical optical system, Cambridge tracts, no. 25, Cambridge, 1928.] The curves are here specified in principle by giving the rectangular coordinates of points on them as functions of polar coordinates ρ, ϑ in the plane of the exit pupil. These functions are then expanded as Fourier series with respect to ϑ and, in a sense, this particular mode of expansion seems to be the main formal difference between the present and Steward's treatment, as far as ordinary optical systems are concerned. The series are approximated by their first three terms, that is, primary aberrations only are considered, though the inclusion of higher order terms is promised for a subsequent paper. The characteristic curves corresponding to the primary aberrations of electron optical systems are treated in some detail, as are also those of ray bundles of large aperture but small field. The paper concludes with 23 examples of characteristic curves, beautifully produced in the form of photographs of patterns on the cathode ray screen of an analogue computer. *H. A. Buchdahl* (Princeton, N.J.)

624:

Blank, Albert A. Axiomatics of binocular vision. The foundations of metric geometry in relation to space perception. J. Opt. Soc. Amer. **48** (1958), 328–334.

The author develops a system of axioms for the geometry of the space of binocular vision, discussing the visual relationship between the apparent distances and the alignment of luminous points in space seen against a dark background. The geometry differs from the usual mathematical type, owing to the fact that the addition of a new point might vary the relationship between previous points. The author is not certain that a one-to-one correspondence can be established between visual and physical space. *M. Herzberger* (Rochester, N.Y.)

625:

Laudet, M. Contribution à l'étude du calcul numérique des champs et des trajectoires en optique électronique des systèmes cylindriques. Ann. Fac. Sci. Univ. Toulouse (4) **20** (1956), 111–230 (1957).

The scope and method of presentation of this review article put it practically in the class of a textbook on numerical calculations in electron optics. The first part deals with the application of the relaxation method to the determination of electromagnetic fields in problems commonly encountered in electrostatics or magnetostatics. The second part is restricted to cylindrical systems having a symmetry plane. The particle trajectory is first determined to a first order approximation. The aberrations are then calculated by the method of successive approximations introduced by Durand for axial symmetry problems [Rev. Optique **33** (1954), 617–629; MR **16**, 652]. The problem of the electrostatic lens formed by three slits in three parallel conducting planes is studied in some detail. *J. E. Rosenthal* (Passaic, N.J.)

626:

Kruskal, M.; and Tuck, J. L. The instability of a pinched fluid with a longitudinal magnetic field. Proc. Roy. Soc. London. Ser. A. **245** (1958), 222–237.

Als Fortsetzung einer Arbeit von M. Kruskal und M. Schwarzschild [dieselbe Proc. **223** (1954), 348–360; MR **15**, 914] wird die Frage der Stabilität eines zusammengezogenen Plasmas, in dem ausser dem vom Strom verursachten azimutalen Magnetfelde ein longitudinales Magnetfeld wirksam ist, untersucht. Den Ausgangspunkt der Rechnungen bilden erstens die vier Maxwellschen Gleichungen, die Bewegungsgleichung

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B}+\varepsilon\mathbf{E}-\nabla p, \tag{1}$$

die Kontinuitätsgleichung

$$\nabla\cdot(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t} \tag{2}$$

und ausserdem die Zusammenhänge

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0 \text{ und } \frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt}. \tag{3}$$

Die erste Gleichung in (3) enthält die Hypothese, dass die Störungen genug langsam verlaufen und die zweite, dass man einfach diese adiabatische Gleichung anwenden kann. Benutzt wird das MKS-System und alle Symbole haben die gewohnte Bedeutung. Im angrenzenden Vakuum gelten nur die Maxwellschen Gleichungen mit $\mathbf{j}=0$ und $\varepsilon=0$.

Zuerst werden aus den Grundgleichungen die Grenzbedingungen an der Plasmaoberfläche hergeleitet, dann werden die Bezeichnungen abgeändert, in dem jetzt die Symbole die kleinen Änderungen der fraglichen physikalischen Grössen bedeuten. Die Gleichungen werden damit linearisiert. Weiter wird angenommen, dass in einer elementaren Lösung jede Grösse den Faktor $\exp(im\theta+ikz+\omega t)$ enthält. Endlich werden einige Veränderliche eliminiert, bis man eine Differentialgleichung vom Besselschen Typ für p als Funktion von r erhält. Auf diesem Wege werden alle Grössen mit Hilfe der Besselschen Funktion erster Art und m-ter Ordnung ausgedrückt. Analog erhält man im angrenzenden Vakuum diese Grössen mit Hilfe von Hankelschen Funktionen. Endlich folgt die charakteristische Gleichung, welche für eine elementare Lösung einen Zusammenhang zwischen m, k und ω angibt. In erster Linie interessant sind unstabile Störungen, für die also ω, oder die dazu proportionale Grösse $w=r_0\omega/s$, einen realen und positiven Teil besitzt.

Die Fälle der Stabilität und Instabilität werden ausführlich besprochen und am Ende der Arbeit werden die Resultate auch graphisch dargestellt. Das wichtigste Ergebnis ist, dass das longitudinale Magnetfeld auf kurzwellige Störungen stabilisierend wirkt, nicht jedoch auf alle solche von grosser Wellenlänge. *T. Neugebauer* (Budapest)

627:

Gold, Louis. Oscillations in a plasma with oriented (D.C.) magnetic field. J. Electronics Control **4** (1958), 409–416.

Ziel der vorliegenden Untersuchung ist die Frequenzen der in einem Plasma – in dem ein von dem Gleichstrom herrührendes elektrisches ($\mathbf{E}=E_x$) und ein dazu willkürlich orientiertes magnetisches Feld ($\mathbf{B}$) vorhanden ist — auftretenden Schwingungen zu berechnen. Den Ausgangspunkt der Untersuchung bildet die aus dem Lorentzschen Kraftansatz folgende Bewegungsgleichung

$$\frac{d}{dt}(m\mathbf{v}) = q\mathbf{E}+q\mathbf{v}\times\mathbf{B} \tag{1}$$

und ausserdem die Gleichung, welche die gegenseitige Wechselwirkung der unbeweglich gedachten Ionen und Elektronen beschreibt:

$$\frac{dE}{dt} = -\frac{1}{\varepsilon_0}(\rho_+ v - F). \tag{2}$$

ρ_+ bedeutet die von den positiven Ionen verursachte Ladungsdichte und F die Stromdichte; alle anderen Symbole haben die gewohnte Bedeutung. Aus den aus der Vektorgleichung (1) folgenden drei Gleichungen erhält man eine Differentialgleichung dritter Ordnung für v_x, aus der weiter durch nochmalige Differentation und Berücksichtigung von (2) man E eliminieren kann. Diese Gleichung wird dann mit Hilfe des Ansatzes

$$v_x = A\cos(\omega_r t-\delta)+F/\rho_+ \tag{3}$$

gelöst, wobei das Resultat

$$\omega_r^2 = \tfrac{1}{2}[\omega^2 \pm \sqrt{(\omega^4-4\omega_p^2\omega_x^2)}] \tag{4}$$

folgt. ω_p ist die Plasmafrequenz $q\rho/m\varepsilon_0$, ω_c die Zyklotronfrequenz qB/m, $\omega^2=\omega_c^2+\omega_p^2$ und ω_x ist qB_x/m. Für den Fall $\mathbf{E}\perp\mathbf{B}$, also $\omega_x=0$ folgt aus (4) das bekannte Resultat $\omega_r^2=\omega^2=\omega_c^2+\omega_p^2$, für $\mathbf{E}\|\mathbf{B}$, also $-\omega_x=\omega_c$ folgt $\omega_r=\omega_c$ bzw. ω_p und für eine schiefwinklige Orientation der Feldstärken folgen ebenfalls zwei Frequenzen, in denen jedoch ω_c und ω_p miteinander gekoppelt vorkommen. Für die Lösung muss jetzt statt (3) der Ansatz $v_x=A_1\cos(\omega_1 t-\delta_1)+A_2\cos(\omega_2 t-\delta_2)+F/\rho_+$ benutzt werden. Die sich auf dieses Problem beziehenden sehr langwierigen Rechnungen werden im zweiten Anhang durchgeführt. Die bis jetzt unberücksichtigte Stossdämpfung wird für den speziellen Fall $\mathbf{E}\|\mathbf{B}$ besprochen. Es sei nur noch bemerkt, dass nach dem Verfasser die angegebene Lösung eigentlich nur die des linearisierten Problems ist, weil man ja in (3) eigentlich $F=F_0+F_1(t)$ schreiben müsste, wo jedoch $F_1(t)$ nur eine kleine periodische Fluktuation bedeutet. *T. Neugebauer* (Budapest)

628:

Sturrock, P. A. A variational principle and an energy theorem for small-amplitude disturbances of electron beams and of electron-ion plasmas. Ann. Physics **4** (1958), 306–324.

From the author's abstract: "Small-amplitude disturbances of electron beam and electromagnetic field

configurations may be conveniently represented by the electromagnetic potentials of the field perturbation and the displacement (or 'polarization') vector representing the perturbation of the electron assembly. With this choice of dynamical variables, it is possible to establish a Lagrangian function for small-amplitude disturbances, in which both field and particles are described by 'field-like' variables. This derivation is carried through in relativistically covariant form." In deriving an energy-momentum tensor for the disturbances from the canonical energy-momentum tensor, the author argues that the usual symmetrization is not possible; a weaker condition is imposed, viz., that it must contain the Maxwell stress tensor and must be expressible in terms of the fields rather than the potentials.

F. Rohrlich (Baltimore, Md.)

629:

Drummond, J. E. Basic microwave properties of hot magnetoplasmas. Phys. Rev. (2) **110** (1958), 293–306.

The usual conductivity tensor of a uniform plasma in a uniform, static magnetic field (along the z axis) is generalized to include four effects of random motion of plasma electrons. These effects are due to (1) the shape of the radio-frequency electric field strength expressed by $\nabla\times\nabla\times E\neq 0$, (2) the partial spanning of a wavelength by an electron cyclotron orbit, and (3) the possible variation of the radio-frequency electric field lines along the z axis of electron drift. The effects resulting cause diffusion damping, the existence of nonzero (x, z), (y, z), (z, x) elements in the conductivity tensor, large changes of the effective plasma density, and phase changes in the conduction current density. These results are applied to evaluation of the index of refraction of microwave signals propagating normal to the magnetic field. The existence of unusual transmission bands is predicted for very dense, hot plasmas. (Author's summary)

K. C. Westfold (Pasadena, Calif.)

630:

Hertweck, F.; und Schlüter, A. Die "adiabatische Invarianz" des magnetischen Bahnmomentes geladener Teilchen. Z. Naturf. **12a** (1957), 844–849.

To explore the contention that the non-relativistic magnetic moment of the motion of a charged particle is approximately constant, the authors treat the case of a particle moving in a uniform field of time varying intensity. In the particular instance of a sudden transition from one near-constant intensity to another, they find that the difference in moment decreases at least exponentially in h/α if α/h tends to zero, where α represents the relative rate of change of the magnetic field and h denotes the gyro-frequency. *E. Pinney* (Berkeley, Calif.)

631:

Moses, H. E. A spinor representation of Maxwell's equations. Nuovo Cimento (10) **7** (1958), supplemento, 1–18.

Maxwell's equations can be written in the following form

$$\text{(M)}\qquad i\frac{\partial}{\partial t}\psi=H_0\psi-4\pi\Phi.$$

Here H_0 is the Dirac Hamiltonian with the mass term being put equal to zero; ψ and Φ are column symbols with the components $(0, H_x-iE_x, H_y-iE_y, H_z-iE_z)$, and (ρ, J_x, J_y, J_z), respectively. (A real time coordinate is used throughout.) The author now shows that if we generate a Lorentz-transformation $x^{i'}=\sum a_k{}^i x^k$, we can restore (M) to its original form provided $\psi'=S\psi$, $\Phi'=T\Phi$, $\psi_0'=0$, $S=\alpha^i T\alpha^{i'}$, $T_{ij}=\lambda a^i{}_j$, λ being arbitrary. This rule holds for proper Lorentz-transformations and for space and time reversals. For time reversals, space reflections and space inversions, the linear relationship between ψ and ψ' has to be changed to $\psi'=S\psi^*$ (asterisk means complex conjugate). The relation between longitudinal gauges and ψ_0 is briefly discussed.

N. L. Balazs (Chicago, Ill.)

632:

Osterberg, Harold. Propagation of plane electromagnetic waves in inhomogeneous media. J. Opt. Soc. Amer. **48** (1958), 513–521.

Consider a medium in which the dielectric constant and the electric conductivity are functions of the z coordinate. The author investigates the propagation of plane electromagnetic waves in this direction, obtaining a Riccati differential equation which he solves exactly. Two new parameters, having some physical significance, essentially determine the propagation laws.

Interesting results relating to reflectance and transmittance are derived. *I. Stakgold* (Washington, D.C.)

633:

Pham, Mau Quan. Induction électromagnétique singulière. C. R. Acad. Sci. Paris **246** (1958), 2734–2737.

634:

Cade, R. The charge density near a sharp point on a conductor. Proc. Cambridge Philos. Soc. **53** (1957), 870–877.

The author examines attempts to confirm a conjecture that a surface charge on a conductor is unbounded near a convex sharp point and approaches zero near a concave point. He finds that the problem has not been solved. He goes on to give a proof which has a fair degree of generality, and he gives reasons why he does not expect to arrive at a complete, unique conclusion.

V. M. Papadopoulos (Providence, R.I.)

635:

Chadwick, P.; and Sneddon, I. N. Plane waves in an elastic solid conducting heat. J. Mech. Phys. Solids **6** (1958), 223–230.

Plane harmonic waves are discussed for a conducting elastic solid satisfying the Duhamel-Neumann equations of state. Typical wave velocities and attenuation coefficients are found. *J. W. Craggs* (Newcastle-on-Tyne)

636:

Bomze, Josef. Elektrische Elementarwellen. I. Ein Beitrag zur Kinematik der elektrischen Elementarladungen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **166** (1957), 77–109.

Ausgehend von den d'Alembertschen Lösungen werden mit Hilfe eines die Punktladung in ihrer Bewegung begleitenden Dreibeins die retardierten (Liénard-Wiechertschen) und die avancierten Potentiale einer bewegten Punktladung abgeleitet und aus diesen, über den Hertzschen Vektor, die Feldgrößen $\mathfrak{D}$ und $\mathfrak{H}$. Die Differentialgleichung der elektrischen Elementarwellen (elektrischen Erregungslinien) wird für t=const (Momentaufnahme) aufgestellt, und durch Einführung des die Ladung in ihrer Bewegung begleitenden Dreibeins gelingt es, diese, im Falle der ebenen Bewegung der Punktladung, in einfacher Weise zu lösen.

Die Bewegung der Gesamtheit der Tangentenvektoren an die elektrischen Erregungslinien im Quellpunkt wird

als Eigenbewegung der Punktladung gedeutet und ihre Differentialgleichung aufgestellt. Diese hängt weder vom Vorzeichen der Ladung noch vom Zustand des Energieaustausches, in dem sich die Ladung befindet, ab. Die Differentialgleichung der Eigenbewegung wird für den Fall der ebenen Bewegung der Elementarladung integriert, und es wird gezeigt, daß jede nichtgeradlinig bewegte Ladung eine Eigendrehung um eine zur Bahnebene senkrechte Achse ausführt. *Zusammenfassung des Autors*

637:

Miranker, W. L. The reduced wave equation in a medium with a variable index of refraction. Comm. Pure Appl. Math. **10** (1957), 491–502.

In this paper a result of Rellich for the reduced wave equation $\Delta u+h^2u=0$ is extended to the case where h is a variable function which, however, tends sufficiently rapidly to a constant k near infinity: $h^2-k^2=O(|x|^{\mu})$, $\mu>2$. If, in addition, the overall deviation of h from k is less than 4π, in the sense of

$$\iiint |h(x)^2-k^2|\frac{dV}{|x|}<4\pi,$$

the author shows that the only bounded solution of the above equation which satisfies the Rellich growth condition

$$\iiint_{|x|\leq R}|u^2|dV=o(R)$$

is identically zero. Several other related results are given.

In the last section it is shown that if in the condition near infinity, μ is greater than 3, but without any overall restriction on h, then the Dirichlet and Neumann problems in any exterior domain have at most one solution satisfying the radiation condition.

P. D. Lax (New York, N.Y.)

638:

Pekar, S. I. The theory of electromagnetic waves in a crystal in which excitons are produced. Soviet Physics JETP **6** (1958), 785–796.

Es wird die Frage besprochen, dass wenn in einem Kristall Exzitonen in allgemeinsten Sinne des Wortes vorhanden sind (also Anregungen die im Kristall beweglich sind), die gewohnten Gleichungen der Kristalloptik dann für Frequenzen in der Nähe der Absorptionsfrequenzen dieser Exzitonen abgeändert werden müssen. Statt des bekannten klassischen Zusammenhanges zwischen elektrischer Feldintensität und dielektrischer Verschiebung $\mathbf{D}=\varepsilon\mathbf{E}$ erhält man nämlich jetzt $\mathbf{D}=\varepsilon'\mathbf{E}+4\pi(\mathbf{P}_1+\mathbf{P}_2)$, wo $\mathbf{P}_1$ und $\mathbf{P}_2$ eben von der Polarisation dieser Exzitonen herrühren. Zwischen dieser zusätzlichen Polarisation und der elektrischen Feldintensität besteht jedoch nicht der gewohnte lineare Zusammenhang (weil ja die Exzitonen eben vom einfallenden Lichte verursacht werden), sondern eine Differentialgleichung vom Typ

$$(1)\qquad [i\hbar\partial/\partial t-\mathscr{E}(-i\nabla)]\dot{\mathbf{P}}_1=ia\mathbf{E}_{\perp}$$

und eine analoge Gleichung für $\mathbf{P}_2$. $\mathscr{E}$ ist hier die Energie des Exzitonenzustandes und a ist ein Tensor zweiter Stufe. Ausserdem bestehen auch die "idealen" Maxwellschen Gleichungen und ausgehend von allen diesen Gleichungen baut der Verfasser eine neue Kristalloptik auf. Sein wichtigstes Resultat ist, dass in einem Kristall nicht nur zwei verschiedene Wellen gleicher Frequenz sich in einer Richtung fortpflanzen können, sondern mehrere, die jedoch verschiedene Brechungsindizes besitzen, was auch in einem kubischen Kristall (der optisch isotrop ist) der Fall ist. Alles das bezieht sich jedoch nur auf Frequenzen, die im Gebiete der Exzitonenabsorption liegen. Besprochen werden die im rhombischen, im tetragonalen und im regulären System auftretenden Verhältnisse und ausserdem in einem gesonderten Paragraphen die von Kristalloberflächen verursachten Erscheinungen. Zuletzt wird noch auf eine theoretische Möglichkeit aufmerksam gemacht, nach der auch rein longitudinale elektrische Wellen in einem Kristall auftreten könnten, die jedoch den Kristall nicht verlassen können.

Die Besprechung der Möglichkeit eines experimentellen Nachweises der erhaltenen Resultate bleibt einer weiteren Veröffentlichung vorbehalten. Die Abweichungen von den bekannten Gesetzen der Kristalloptik müssen jedenfalls sehr gering und unbedeutend sein.

T. Neugebauer (Budapest)

639:

Wait, James R. Excitation of surface waves on conducting, stratified, dielectric-clad, and corrugated surfaces. J. Res. Nat. Bur. Standards **59** (1957), 365–377.

The expression is found for the field of an electric dipole located over a flat surface of specified surface impedance. It is shown how the nature of the radiated field depends on the argument of the (complex) impedance. For an argument greater than $\pi/4$, the surface acts as a guide, this effect being most marked when the impedance is purely imaginary.

This is a paper in which the connection between various types of surface wave, with phase velocities both less than and greater than the velocity of light, is brought out. It now appears reasonable to describe a surface wave by the physical nature of the surface instead of referring to the name of the original discovery. However, it is not made clear how the surface impedance boundary condition may be related to a more realistic one where the penetration into the ground is taken into account.

V. M. Papadopoulos (Providence, R.I.)

640:

***Keller, Joseph B. A geometrical theory of diffraction.** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. 8, pp. 27–52. For the American Mathematical Society: McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. v+153 pp. $7.50.

The purpose of this article is to show that geometrical optics can be modified to include diffraction effects. This modification is effected by extending the laws of geometrical optics to include other rays, called diffracted waves. This modification is carried out in two ways. The first is an explicit form in which the author enumerates the different situations in which diffracted waves may occur while the second is based on the Fermat principle. The material discussed is well illustrated and a good bibliography is provided. *A. E. Heins* (Pittsburgh, Pa.)

641:

Keller, Joseph B. Diffraction by a convex cylinder. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-94** (1956), 10 pp. Also: Trans. I.R.E. **AP-4** (1956), 312–321.

The geometrical theory of diffraction proposed previously by the author is used to determine the leading term in the asymptotic expansion, for large wave number, of the fields reflected and diffracted by any convex cylinder. In this theory the concept of a diffracted ray is introduced and an appropriate field is associated with each such ray.

Several special cases are discussed in detail. These include circular and parabolic cylinders and a thick screen with a circular end. The results are compared with solutions obtained by different methods by other authors.

E. Wolf (Manchester)

642:

Silverman, Richard A. Fading of radio waves scattered by dielectric turbulence. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-101** (1957), i+15 pp.

An antenna A emits monochromatic electromagnetic radiation of wavelength λ, which is scattered into a receiving antenna A' at a distance D beyond the geometrical horizon from A, by the dielectric noise in the common volume of the antenna beams; the received radiation is described by a random function $I(t)$ (complex amplitude of the current in the receiving antenna); $I(t)$ is assumed to be a complex stationary gaussian random function and consequently described by its correlation function. The main result is that this function is the product of 2 correlation functions: (a) one representing time variation of the scattered eddies, as seen in a system moving with the local wind velocity, which in the case of atmospheric turbulence is itself a random process; (b) the other, the convective Döppler shifting, due to convection of the dielectric constant patterns that cause scattering by the mean wind and by the large velocity eddies.

J. Kampé de Fériet (Lille)

643:

Zverev, V. A. The influence of the directivity of the receiving system on the average intensity of the signal received as a result of the scattering. Akust. Ž. 3 (1957), 329–336. (Russian)

644:

Bosma, H. Two capacitive windows in a rectangular wave guide. Appl. Sci. Res. B. 7 (1958), 131–144.

By use of Schwinger's integral equation method, the author deduces approximate expressions for the reflection and transmission coefficients of a dominant-mode wave traveling along a rectangular waveguide loaded with two closely spaced capacitive windows. Since the windows are closely spaced, it is necessary to take into account the evanescent modes, in addition to the usual dominant mode, which couple together the two windows. The symmetry of the structure with respect to the incident dominant mode permits a two-dimensional scalarization of the problem.

The mathematical formulation leads to two simultaneous integral equations for the aperture electric fields which, when averaged over the surface of the apertures, yield the reflection and transmission coefficients. The aperture electric fields are expanded in Fourier series. The two kernel functions are approximated by truncating their series expansions. Thus the problem is reduced to a finite system of linear algebraic equations. From a physical viewpoint, truncating the kernels is equivalent to neglecting all but a finite number of the evanescent modes.

The author mentions that B. J. Beukelman has handled this problem by means of Schwinger's variational principle and that the author's calculation resulted in slightly more correct expressions than the ones obtained by Beukelman (unpublished). However, this reviewer wonders whether Schwinger's variational principle was fully applied in this instance, since the nature of the problem is such that the variational principle could yield upper and lower bounds and an iteration procedure for the step by step improvement of the result could be carried out.

C. H. Papas (Pasadena, Calif.)

645:

Chambers, Ll. G. The propagation of constant longitudinal magnetic waves in dielectric filled waveguides. Quart. J. Mech. Appl. Math. 11 (1958), 244–252.

The author discusses the conditions under which CLM, CLE, and TEM waves may exist in a rectangular waveguide loaded with longitudinal dielectric slabs.

C. H. Papas (Pasadena, Calif.)

646:

Caprioli, Luigi. Sulla attenuazione nelle guide circolari con pareti assorbenti. Boll. Un. Mat. Ital. (3) 12 (1957), 526–534.

The effect of lossy walls on the propagation of electromagnetic waves in a pipe of circular section is examined in detail. The author derives the attenuation constant for frequencies close to the cut-off frequencies of individual modes.

V. M. Papadopoulos (Providence, R.I.)

647:

Frank, V.; and Jensen, H. Højgaard. Note on the reciprocity theorem for electrical systems. Appl. Sci. Res. B. 7 (1958), 145–149.

It is shown from Maxwell's equations that if in a general plane d.c. network (of discrete and/or continuous components) a homogeneous algebraic relation of the form

$$C_1(\mathbf{r}, B_0(\mathbf{r}))\cdot\sigma_{\alpha\beta}(\mathbf{r}, B_0(\mathbf{r}))+C_2(\mathbf{r}, B_0'(\mathbf{r}))\cdot\sigma_{\beta\alpha}(\mathbf{r}, B_0'(\mathbf{r}))\equiv 0$$

exists — where $\mathbf{r}$ means two-dimensional vector loci, C_1, C_2 scalar functions, σ_{ij} components of the conductivity tensor, and $B(\mathbf{r})$, $B'(\mathbf{r})$ externally imposed static magnetic induction fields — the same algebraic relation as for σ_{ij} holds for the transconductances G_{ij} associated with $(n+1)$ terminals for this network. The important special case is that of the gyrator where B_0 is a homogeneous field and $\sigma_{\alpha\beta}(B_0)=\sigma_{\beta\alpha}(-B)$ results in $G_{ij}(B_0)=G_{ij}(-B)$. The proof of the generalization to a.c., with complex frequencies, which encompasses the permittivity and inverse permeability tensors $\varepsilon_{\alpha\beta}$ and $(\mu_{\alpha\beta}^{-1})$, as well as $\sigma_{\alpha\beta}$, follows the same line and is sketched.

H. G. Baerwald (Cleveland, Ohio)

648:

Brown, J. Propagation in coupled transmission line systems. Quart. J. Mech. Appl. Math. 11 (1958), 235–243.

A set of transmission lines are coupled at equal intervals by identical networks and the modes of wave-propagation are studied.

G. Kron (Schenectady, N.Y.)

649:

Horn, R. E.; and Fauque, V. G. Synthesis of vector networks. Trans. I. R. E. **EC-6** (1957), 261–265.

The direct relationship between vector equations and matrix synthesis methods for networks is exploited to give a method of network synthesis that proceeds straight from the vector notation. Certain elementary vector networks are used to synthesize the more complex ones usually needed in applications, and result in a closer analogy between the mathematics and the electronics.

V. E. Beneš (Murray Hill, N.J.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 440, 563.

650:

Pauli, Wolfgang. Zur Thermodynamik dissoziierter Gleichgewichtsgemische in äusseren Kraftfeldern. Z. Angew. Math. Phys. **9b** (1958), 490–497.

The use of semipermeable membranes for the derivation of thermodynamical laws is somewhat doubtful, in particular, if it has to be assumed in addition that a chemical reaction can be inhibited. It is here shown, for the case of the chemical equilibrium $J_2 \rightleftarrows 2J$, that the different behavior of both components in a suitably chosen external field suffices to derive these laws. As the iodine molecules are diamagnetic and the atoms are paramagnetic, an external magnetic field has a different effect on both chemical potentials. This permits one to derive the known relations for chemical equilibrium. The result is applied, in particular, to the case of ideal gases. *N. G. van Kampen* (Utrecht)

651:

Thomas, P. H. Some conduction problems in the heating of small areas on large solids. Quart. J. Mech. Appl. Math. **10** (1957), 482–493.

The initial temperature of a semi-infinite solid $x \geqq 0$ is taken to be zero. Into a circular region of the face $x=0$ linear heat transfer takes place from a hot gas, while the rest of the face is cooled by gas at temperature zero. Thus the temperature function θ satisfies a condition at $x=0$ of the form $H\theta - K\theta_x = Q(r)$, $t>0$, where $Q(r)=Q_0$ when $r<R$ and $Q(r)=0$ when $r>R$, $(r^2=y^2+z^2)$. The heat equation is solved under those conditions and a convenient formula is obtained for the temperatures $\theta_0(t)$ at the center $r=0$, $x=0$ of the circular region. Then the dependence of $\theta_0(t)$ upon the radius R and other parameters is investigated. Some results are shown graphically, results that are of interest in connection with fire hazards. Similar studies are made of steady temperatures on the face $x=0$ caused by an external point source of radiation, and of transient temperatures in thin slabs with a circular region or a strip exposed to hot gas. Fourier or Hankel transforms are employed to solve some forms of the heat equation that occur. *R. V. Churchill* (Ann Arbor, Mich.)

652:

Gor'kov, L. P. Stationary convection in a plane liquid layer near the critical heat transfer point. Soviet Physics. JETP **6** (1958), 311–315.

For certain values of a downward-directed temperature gradient in a plane horizontal liquid layer, pure thermal conduction is replaced by convection with certain characteristics in a stationary mode. The critical gradient at which convection occurs has been studied in the literature [A. Pellew and R. Southwell, Proc. Roy. Soc. London Ser. A **176** (1940), 312–343; MR **2**, 266]. This paper gives the determination of the amplitude of the motion above, but close to, the critical value. The problem is investigated by means of Fourier expansions of the variables. It is found that in a hypothetical case of a liquid with two free surfaces the strength of stationary thermal convection in a plane liquid layer for modes near the critical point is proportional to the square root of a parameter which characterizes supercriticality. *L. N. Tao* (Chicago, Ill.)

653:

Poots, G. Heat transfer by laminar free convection in enclosed plane gas layers. Quart. J. Mech. Appl. Math. **11** (1958), 257–273.

This paper deals with the heat transfer problem by free laminar convection in a (two dimensional) closed rectangular cell inclined at an angle to the vertical. The boundary conditions are assumed to be two different temperatures at the walls and a linear temperature distribution or a perfect insulation at the border strips. Based on the governing equations derived by Batchelor [Quart. Appl. Math. **12** (1954), 209–233; MR **16**, 299], the problem is solved numerically by assuming the solutions to be in the form of orthogonal polynomials. The calculated results agree with previous formulas correlated from experimental data. Calculations also indicate possible existence of an isothermal core of constant vorticity in the cavity, postulated by Batchelor, as the product of the Prandtl and Grashof numbers becomes very large. *L. N. Tao* (Chicago, Ill.)

654:

Forster, H. Kurt. Diffusion in a moving medium with time-dependent boundaries. A. I. Ch. E. J. **3** (1957), 535–538.

This concerns heat flow in a medium having known velocity distribution. The author proposes an approximation scheme based on the usual diffusion equation for a medium at rest, but does not establish its convergence to the exact solution. He gives some results of an application of this theory to study the heat transfer into a spherical cavity having radial motion. He discusses the results in detail. *E. Pinney* (Berkeley, Calif.)

655:

Millsaps, Knox; and Pohlhausen, Karl. The laminar free-convective heat transfer from the outer surface of a vertical circular cylinder. J. Aero. Sci. **25** (1958), 357–360.

The boundary-layer problem that treats the laminar free-convective fluid motion produced by a heated vertical circular cylinder for which the thermal distribution on the outer surface varies linearly with the distance from the leading edge admits a similarity transformation that reduces the momentum and energy equations to two ordinary equations. The exact solutions to the ordinary differential system are obtained by numerical methods for parametric values of the Grashof and Prandtl numbers. A comparison is given between the exact solutions and the approximations that are obtained using a parabolic assumption for the thermal profile. (Author's summary.) *M. Lister* (State College, Pa.)

656:

Miranker, W. L. A free boundary value problem for the heat equation. Quart. Appl. Math. **16** (1958), 121–130.

Let $u(x, t)$ satisfy the heat equation $u_t = \alpha^2 u_{xx}$ in a domain $t>0$, $0<x<R(t)$, and let that function and the function $R(t)$ satisfy these boundary conditions: $R(0)=A$, $u_x(0,t)=-G(t)<0$, $u[R(t), t]=T_c$, $u(x,0)=F(x)\geqq T_c$ wehn $0\leqq x\leqq A$, and $u_x[R(t), t]=B-CR'(t)$ $(B\geqq 0, C>0)$. This is a Stefan problem in the two unknown functions $u(x, t)$ and $R(t)$. The existence of its solution is established here with the aid of a procedure used in similar problems by I. I. Kolodner [Comm. Pure Appl. Math. **9** (1956), 1–31; MR **19**, 285]. A nonlinear integrodifferential equation in the boundary function $R(t)$ is set up. The existence of a solution of that equation, satisfying certain conditions, is

established. A mapping, associated with that equation, in a certain Banach space is introduced to help establish the existence. The author shows how the existence of a solution of the Stefan problem follows from the existence of the function $R(t)$.

R. V. Churchill (Ann Arbor, Mich.)

657:

Spalding, D. B. Approximate solutions of transient and two-dimensional flame phenomena: constant-enthalpy flames. Proc. Roy. Soc. London. Ser. A. **245** (1958), 352–372.

The behavior of the transient flames resulting from contact of semi-infinite burnt and unburnt gas masses, contact of unburnt gas with an adiabatic catalyst, immersion of a finite slab of unburnt gas in a large mass of hot gas, and immersion of a finite slab of unburnt gas in a larger mass of hot gas is investigated with the assumptions that the volumetric reaction is a function of temperature alone and that the gas medium is either at rest or in uniform motion. The problems are solved approximately by the integral methods of boundary-layer theory. In the first two cases three different temperature profiles in the flame (linear, sinusoidal, and second order) are adopted. Comparisons with exact solutions show that for the first case the second order profile has the best agreement and that the linear profile is best in the second case. The other two cases are studied by the linear profile only. It is claimed that the accuracy of the method is of the order of 20%.

L. N. Tao (Chicago, Ill.)

658:

Jain, B. S. Internal ballistics of a leaking gun. Proc. Nat. Inst. Sci. India. Part: A. **22** (1956), 324–335 (1957).

The leakage of propellant gas during travel of the projectile in the bore of a traditional gun has usually been ignored in the first approximations to the phenomena of interior ballistics. Leakage may be significant in a worn gun or in the case of imperfect obturation. Leakage is of major importance during open-nozzle in the case of recoilless guns. Rephrased, it is essentially the motive power in rocker propulsion. The relevant equations were established by Corner [Proc. Roy. Soc. London. Ser. A. **188** (1947), 237–255; Theory of the interior ballistics of guns, Wiley, New York, 1950; MR **12**, 213]. For the case in which burning is proportional to pressure, he gave a simple approximate solution based on semi-empirical grounds. V. R. Thiruvenkatachar and N. S. Venkatesan [Proc. Nat. Inst. Sci. India **19** (1953), 829–837] obtained a reasonably accurate theory by expanding in series of Ψ the leakage parameter, with a solution worked out for $\theta=0$ (tubular powder), dropping terms in Ψ above the first power. This author further explores the mathematical theory. Even for $\theta=0$, the term in Ψ^2 is found to be non-negligible. For recoilless guns he recommends numerical methods. Adopting f, the fraction of ballistic size propellant remaining at time t as independent variable, the author obtains the coefficient of the first power of Ψ in closed form (for $\theta=0$, as an elliptic integral) and shows how to compute the coefficient of the second power.

A. A. Bennett (Carbondale, Ill.)

659:

Kapur, J. N. The internal ballistics of a recoil-less high-low pressure gun. Appl. Sci. Res. A. **6** (1957), 445–466.

Corner [op. cit., review #658 above; and J. Franklin Inst. **246** (1948), 233–248] has studied the internal ballistics of recoilless and of high-low pressure guns (the latter have two chambers in series). The present paper extends the author's previous studies for the isothermal model of the recoilless high-low pressure gun [Proc. Nat. Inst. Sci. India. Part A **23** (1957), 312–321; MR **19**, 1014]. Effects of degressive and of progressive grain shape become more pronounced in this sort of gun. Partially non-isothermal models are studied, including one in which temperature is constant in the first chamber, and one in which pressure, gas mass, and internal energy increase in the first chamber, while temperature decreases there but remains greater than a limiting value. Some numerical values are secured from elaborate derived formulas.

A. A. Bennett (Carbondale, Ill.)

660:

Kapur, J. N. Unified theory of internal ballistics. Quart. J. Mech. Appl. Math. **11** (1958), 98–111.

The author discusses the equations for interior ballistics under general assumptions designed to include as special cases the "orthodox" gun, and the solid-fuel rocket. While the equations offer nothing new in these two familiar special cases, they remain applicable to many intermediate types of weapons such as recoilless guns, and high-low pressure (double-chambered) guns, and other such forms which have been currently handled independently, and hitherto often only in the isothermal case. The integration, even in relatively special particular cases, continues to present difficulties.

A. A. Bennett (Carbondale, Ill.)

661:

***Kapur, J. N. The internal ballistics of a high-low pressure gun 'after burnt'.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 209–218. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

The author investigates the internal ballistics of a high-low (double-chambered) gun after all-burnt, although currently this phenomenon occurs near the muzzle. If, however, the propellant be consumed early in the first chamber, an energy term hitherto neglected becomes significant, and older theories might yield an error in muzzle velocity of as much as 30 per cent. This article applies to nonisothermal burning, and does not ignore heat losses, and covolume terms.

A. A. Bennett (Carbondale, Ill.)

662:

Kapur, J. N. The equivalent charge method in the general theory of composite charges. II. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 469–482.

The equivalent charge method for dealing with the internal ballistics for composite charges seeks to determine the nominal uniform charge which would produce the same ballistic effects both during burning and after all-burnt, as does a given composite charge. The problem involves so many parameters that one might well question whether there might not be many different choices for an equivalent charge, in a given gun and with a given composite charge. The author points out that, on the other hand, there may exist no equivalent charge if one insists upon keeping constant with respect to time the parameters usually assumed to be constant. In practice time-varying corrections can be introduced which with suitable average values yield satisfactory results. The difficulties rise prior to all-burnt. The theory of composite charges has been extended to the theoretically and experimentally more plausible linear law of burning.

A. A. Bennett (Carbondale, Ill.)

663:
Sodha, M. S.; and Jain, V. K. Internal ballistics of a high velocity light gas gun. Appl. Sci. Res. A. **7** (1958), 351–356.

A gun proposed by Crozier and Hume [J. Appl. Phys. **28** (1957), 892–894] achieves a high muzzle velocity by the introduction of energy absorbers and a piston, separating the basic powder chamber from a quantity of light gas immediately in the rear of the projectile. This brief paper gives the equations of internal ballistics applicable to such a weapon. *A. A. Bennett* (Carbondale, Ill.)

664:
Jain, V. K.; and Sodha, M. S. On the internal ballistics of a supergun (Hochdruckpumpe). Appl. Sci. Res. A. **7** (1958), 369–374.

An often recurring idea is to progressively boost the velocity of a projectile during passage along the bore, by placing successive auxiliary energy sources along the bore. Such a plan has been tried unsuccessfully many times for military weapons, where the energy was to be provided by powder charges, or for an electro-magnetic gun by successively energized magnetic rings. These authors, without reference to earlier such super-guns, state "The most spectacular gun developed by Germany was ··· a 15 cm gun made up of 40 sections, with 28 powder chambers distributed along the bore, so as to give successive explosions to boost the projectile along the barrel." This design was never realized. The authors provide the equations of internal ballistics for such a gun. An analytic solution is found under certain simplifying assumptions. *A. A. Bennett* (Carbondale, Ill.)

QUANTUM MECHANICS

See also 477, 529, 724, 725.

665:
Golden, S. A formal theory of quantum classification. I. Nuovo Cimento (10) **5** (1957), supplemento, 540–567.

The author develops a formalisation of the interpretation of projective operators in quantum theory, using the methods of Boolean algebra and introducing certain classification operators τ, such that any given idempotent operator is uniquely represented in the form $\pi+\tau$, where π is hermitian and idempotent. Applications are given to operators representing conjugate variables and further applications are promised.

G. Temple (Oxford)

666:
Białynicki-Birula, I. Schwinger's functional without any anticommuting external sources. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1127–1129, XCIII. (Russian summary)

The author discusses a functional of external sources from which propagators may be obtained by differentiation. The equations satisfied by the propagators are also discussed. *A. H. Taub* (Urbana, Ill.)

667:
Bürgel, Bruno. Über die Lösungen der radialen Schrödinger-Gleichung bei beliebigem Spin. Helv. Phys. Acta **30** (1957), 395–406.

Results obtained by Newton [Phys. Rev. **100** (1955), 412–428; MR **17**, 614] are generalised to include the consideration of the radial part of an undor of the nth rank. It is shown that the solutions of the relevant Schrödinger equation vanishing at the origin form a n-dimensional continuum in the $2n$-dimensional space of the $y_1, \cdots, y_n$, $y_1', \cdots, y_n'$ (the y_i are the undor components, and the y_i' their derivatives with respect to r).

D. ter Haar (Oxford)

668:
Viguier, Gabriel. L'équation de Schrödinger pour un oscillateur harmonique linéaire. Bul. Inst. Politech. Iaşi **4** (1949), 226–232. (1 insert)

The article deals with the relationship between the Schroedinger equation for the harmonic oscillator and the Riccati differential equations. In particular, it shows that by suitable transformation the Schroedinger equation for the harmonic oscillator can be transformed into the Riccati type differential equation, and an analogue is found in this context for the uaqntization in the Schroedinger equation. *M. J. Moravcsik* (Livermore, Calif.)

669:
Schmidt, Helmut. Über die invarianten und kausalen Lösungen der Differentialgleichungen $(\Box\mp\kappa^2)\,\psi(\tau,\ell)=\delta^4(\tau,\ell)$. Z. Physik **151** (1958), 365–374.

Versteht man unter $\delta^4(\tau,\ell)$ Dirac's δ-Funktion im raumzeitlichen Kontinuum und unter $\psi(\tau,\ell)$ die Wellenfunktion eines Teilchens der Masse $m=\kappa\hbar/c$, so gilt im Vakuum die homogene Wellengleichung

$$(*)\qquad (\Box-\kappa^2)\psi(x)=0.$$

Demgegenüber werden durch

$$(**)\qquad (\Box+k^2)\psi(x)=0.$$

Wellen beschrieben, die sich mit Überlichtgeschwindigkeit ausbreiten (die Gruppengeschwindigkeit, welche die Ausbreitungsgeschwindigkeit von Wellenpaketen bestimmt, ist größer als c). Zwei Weltpunkte $x=(\tau,\ell)$ und $x'=(\tau',\ell')$ heißen äquivalent, wenn es eine homogene eigentliche Lorentztransformation gibt, welche x in x' überführt. Gilt dann $\psi(x)=\psi(x')$, so wird $\psi(x)$ invariant genannt. Zu (*) gehören zwei unabhängige invariante Lösungen, deren Darstellung durch komplexe Integrale bekannt ist. Für eine invariante Lösung von (**) gewinnt Verfasser die Darstellung

$$G^1(x)=-\tfrac{1}{2}(2\pi)^{-3}\int d^4k\,\varepsilon^{ikx}\delta(k^2-\kappa^2),\quad d^4k=d^3\mathfrak{k}\cdot dk_0.$$

Im inhomogenen Falle

$$(***)\qquad (\Box-\kappa^2)\psi(\tau,\ell)=\delta^4(\tau,\ell)$$

läuft der Integrationsweg in der komplexen k_0-Ebene von $-\infty$ bis $+\infty$ und führt im wesentlichen auf Lösungen, die als aus- bzw. einlaufende Wellen interpretiert werden können ($k^2=k_\mu k_\mu=\mathfrak{k}^2-k_0^2$, $k_\mu=(\mathfrak{k}, ik_0)$). Die Funktion

$$G_{\mathfrak{C}}(\tau,\ell)=-(2\pi)^{-4}\int_{\mathfrak{C}} e^{ikx}(k^2-\kappa^2)^{-1}\,d^3\mathfrak{k}dk_0$$

ist Lösung der homogenen Gleichung (**) für jeden in der k_0-Ebene geschlossenen Weg $\mathfrak{C}$ und Lösung der inhomogenen Gleichung

$$(****)\qquad (\Box+\kappa^2)\psi(\tau,\ell)=\delta^4(\tau,\ell)$$

für jeden Weg C von $-\infty$ nach $+\infty$. In der so gewonnenen Darstellung der Lösungen lassen sich wieder Zerlegungen nach ein- und auslaufenden Wellen angeben. Unter den so gewonnenen Lösungen der Gleichung sind diejenigen

physikalisch bedeutungslos, welche für $t<0$ verschwinden, da sie mit t exponentiell anwachsende Anteile enthalten. Doch gewinnt Verfasser (mit Verwendung eines geeigneten Integrationswegs) im Falle (****) die physikalisch brauchbare kausale (retardierte) Lösung

$$G_R=-(2\pi)^{-2}r^{-1}\int_0^\infty \cos[r\sqrt{(\kappa^2+u^2)}-tu]du,$$

zu welcher auch die entsprechende avancierte Funktion (durch Vertauschen von $-tu$ mit $+tu$) erhalten wird.

M. Pinl (Köln)

670:

Reeves, Hubert. Inequality relations for scattering cross sections. Ann. Physics 3 (1958), 386–396.

A lemma of Erdös and Fuchs [J. London Math. Soc. 31 (1956), 67–73; MR 17, 586] states that if $\varphi(z)=\sum_{n=0}^\infty b_n z^n$ is convergent for $|z|<1$, the inequality

$$\int_{-\alpha}^{\alpha}|\phi(|z|e^{i\theta})|^2d\theta\geq\frac{\alpha}{3\pi}\int_{-\pi}^{\pi}|\phi(|z|e^{i\theta})|^2d\theta$$

is obeyed. Choosing an axially symmetric scattering amplitude $g(\theta)$ or an angle function related to it for ϕ, several inequalities between the Legendre coefficients of the scattering amplitude are derived. However the physical significance of these inequalities is quite unclear.

E. C. G. Sudarshan (Cambridge, Mass.)

671:

Feenberg, E.; and Goldhammer, P. Further refinements on the Brillouin-Wigner perturbation procedure. Phys. Rev. (2) 105 (1957), 750–755.

This is a generalization of the modified Brillouin-Wigner perturbation theory [P. Goldhammer and E. Feenberg, same Rev. (2) 101 (1956), 1233–1234], in the case where the perturbation energy operator is a sum of physically different parts. Only first order terms are considered and question of convergence is not studied. Degeneracy case, however, is worked out and the main feature of the modified theory, that is, to give results only in terms of quantities occurring in the standard Brillouin-Wigner theory, is preserved by the generalization. Application to the deuteron problem shows how the introduction of new parameters improves the energy value.

D. Rivier (Lausanne)

672:

Finkelstein, R. Permutation symmetries of generalized beta interactions. Phys. Rev. (2) 109 (1958), 1842–1845.

The author is concerned with second degree forms in the coefficients g_σ and U_σ which are defined below and which are independent of the order in which the spinors, a, b, c, and d entering into their definition, are written. Particular forms are derived for four and eight component spinors. A general interaction may be written as

$$F=\sum_0^{2n} g_\sigma E^{(\sigma)}+\sum_0^{2n} U_\sigma O^{(\sigma)}$$

$$E(a,b,c,d)=\sum_\rho (aE_\rho{}^{(\sigma)}b)(cE_\rho{}^{(\sigma)}d)$$

$$O(a,b,c,d)=\sum_\rho (aE_\rho{}^{(\sigma)}b)(aE_\rho{}^{(\sigma)}Qd)$$

$$Q=\prod_1^{2n}\gamma_i.$$

γ_i are the Dirac matrices, $(aE_\rho{}^{(\sigma)}b)$ is the ρth component of a tensor of rank σ formed from the γ's and the spinors a and b and $(aE^{(\sigma)}Qb)$ is similarly defined.

A. H. Taub (Urbana, Ill.)

673:

Donnert, Hermann. Zur Theorie relativistisch invarianter Spinwellengleichungen. I, II. Acta Phys. Austriaca 11 (1957), 321–376.

In these two papers the author uses the van der Waerden spinor formalism to review the work done on wave equations for particles of non-zero mass and arbitrary spin S. A series of mathematically equivalent wave equations for free particles is discussed. The question of which equations obtain in case electromagnetic forces are present is raised and an answer is proposed, namely those satisfied by wave functions having $2(2S+1)$ linearly independent components. An extensive but not exhaustive bibliography of papers on relativistic wave equations is included.

A. H. Taub (Urbana, Ill.)

674:

Swann, W. F. G. Mass-energy relation in quantum theory. Phys. Rev. (2) 109 (1958), 998–1008.

The author proposes a classical theory of rest-mass defects based on the conception of a nucleon as a number of subparticles, "specks", which obey the conservation of relativistic energy and momentum in a suitably chosen frame of reference. The consequences of these assumptions are then applied to the quantum theory of nucleons which are assumed to be described by an equation derived from the nonrelativistic Schrödinger equation. The justification for using this equation instead of a relativistically invariant one such as the Dirac equation or the Klein-Gordon equation is not made clear.

A. H. Taub (Urbana, Ill.)

675:

Nowak, W.; und Tietz, T. Über eine Vereinfachung der Sommerfeldschen Polynommethode. Ann. Physik (7) 1 (1958), 296–298.

The paper describes a simplification of the Sommerfeld polynomial method for determining the eigenvalues of Schroedinger's equation. A solution of the type $R=yf$ is assumed where f satifies the boundary conditions $R(\infty)=R(0)=0$; and y a differential equation. In many cases, the function y satisfies a differential equation with polynomial coefficients whose eigenvalues are known and whose eigenfunctions can readily be determined. The method is illustrated by several examples.

C. Froese (Vancouver, B.C.)

676:

Süssmann, Georg. Über den Messvorgang. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. (N.F.) no. 88 (1958), 41 pp.

The paper contains a careful discussion of the measurement problem in quantum theory. The usual statistical interpretation of quantum theory due to Born, Jordan and Dirac is admitted and supported against an eventual thesis that the statistical character of quantum theory is only a consequence of interaction between the object to be measured and the measurement apparatus, the state of which is in principle unknown before the measurement.

W. Królikowski (Warsaw)

677:

Nagy, K. L. Relativistic equation for the distinguished component of the state vector. Nuovo Cimento (10) 8 (1958), 32–38.

This is a covariant version of the method of Królikowski and Rzewuski [Nuovo Cimento (10) 3 (1956), 260–275; 4 (1956), 1212–1215; MR 19, 218] in discussing the time dependence of a specific projection of the state vector

of a quantum mechanical system. The equation of motion is an integro-differential equation, which can be reduced to a purely differential equation by introducing a "potential density" (which in turn is characterized by an integral equation). As expected the time development is non-unitary. As an example of the method, the "dressing" of a V particle in the Lee model (T. D. Lee: Phys. Rev. **95** (1954), 1329–1334; MR **16**, 317] is studied.

E. C. G. Sudarshan (Cambridge, Mass.)

678:

Aharonov, Y.; and Bohm, D. On the measurement of velocity of relativistic particles. Nuovo Cimento (10) **5** (1957), supplemento, 429–439.

The authors criticize Dirac's interpretation of the fact that the only eigenvalues of the velocity operator of a Dirac electron are $\pm c$, namely that a strongly localized relativistic particle will possess a very large kinetic energy, on the grounds that the same interpretation might be applied to particles obeying different relativistic wave equations. Analyzing the Dirac electron both by conceptual experiments and by formal manipulation they conclude that, whereas large velocities are a common feature of relativistic particles, the independence of the velocity of the linear momentum, and the commutability of the velocity and the location, are peculiar to the Dirac particle.

P. G. Bergmann (Syracuse, N.Y.)

679:

Everett, Hugh, III. "Relative state" formulation of quantum mechanics. Rev. Mod. Phys. **29** (1957), 454–462.

The measuring apparatus (the observer) and the system being measured (the object-system) are regarded as forming parts of a composite quantum-mechanical system described by a wave function continuous in time. If such a composite system is in a given state, the two subsystems will in general not be in definite states, but to each state of one part one can assign uniquely a corresponding "relative state" to the other part. The observer is regarded as having a memory so that it records observations, i.e., quantities characterizing its states and, therefore, under suitable circumstances, the states of the object system. The state of the combined system can be regarded as a superposition, each element of which contains a definite observer state and a corresponding system state. With each succeeding observation the observer state "branches" into a number of different states. In the sequence of observations the measure assigned to a given observed value is shown to be the absolute square of the amplitude of the corresponding element of the superposition. This leads to the same distribution of probabilities as in the usual treatment of measurement in quantum mechanics.

N. Rosen (Haifa)

680:

Wheeler, John A. Assessment of Everett's "relative state" formulation of quantum theory. Rev. Mod. Phys. **29** (1957), 463–465.

The theory of the preceding paper is discussed. A comparison is made between the conventional "external observation" form of quantum mechanics in which the observer is considered as being outside the system being measured, and the conceptual scheme of "relative state" quantum mechanics in which the observer together with the system being measured are regarded as forming a composite quantum-mechanical system.

N. Rosen (Haifa)

681a:

Wiener, N.; and Siegel, A. The differential-space theory of quantum systems. Nuovo Cimento (10) **2** (1955), supplemento, 982–1003.

681b:

Siegel, Armand, and Wiener, Norbert. "Theory of measurement" in differential-space quantum theory. Phys. Rev. (2) **101** (1956), 429–432.

Heuristic presentations continuing the discussions [initiated in Phys. Rev. (2) **91** (1953), 1551–1560; MR **15**, 273] of quantum phenomenology and non-relativistic dynamics in relation to Wiener's theory of Brownian motion.

I. E. Segal (Copenhagen)

682:

Jauch, J. M. Theory of the scattering operator. Helv. Phys. Acta **31** (1958), 127–158.

This is an attempt to put quantum mechanical scattering theory on a rigorous mathematical basis and therefore deserves considerable attention. After a brief review of some of the basic definitions and theorems of a separable Hilbert space $\mathfrak{H}$, a "simple scattering system" is defined as a single-channel physical scattering system characterized by the semi-bounded self-adjoint linear operators H and H_0 whose domains are everywhere dense in $\mathfrak{H}$, and which satisfy the following three conditions. (I) Let $U_t=\exp(-iH_0t)$ $V_t=\exp(-iHt)$; then $\lim_{t\to\mp\infty} V_t^*U_tf=f_\pm$ exists strongly for all $f\in\mathfrak{H}$. (II) The sets $R_\pm$ of all vectors $f_\pm$ satisfy $R_+=R_-=R$. (III) Let N be the orthogonal complement to the subspace spanned by all eigenvectors of H; then $N\subset R$. (This last requirement essentially amounts to $N=R$, since (I) and (II) imply $N\supset R$). The main theorem concerns the existence of the wave operators in a simple scattering system: The limits $\Omega_-=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty e^{-\varepsilon t}V_t^*U_tdt$ and $\Omega_-^*=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty e^{-\varepsilon t}U_t^*V_tdt$ exist in the strong sense everywhere in $\mathfrak{H}$ and are bounded linear operators satisfying $\Omega_-^*\Omega_-=I$, $\Omega_-\Omega_-^*=I-E_M$, where I is the identity and E_M, is the projection into $M=R^\perp$. The same theorem holds for $\Omega_+=\lim_{\varepsilon\to+0}\varepsilon\int_{-\infty}^0 e^{\varepsilon t}V_t^*U_tdt$ and Ω_+^*. The scattering operator is then defined by $S=\Omega_-^*\Omega_+$ and is unitary. It is remarked that the above integral representation for Ω_- could have been replaced by $\Omega_-=\lim_{T\to\infty}T^{-1}\int_0^T V_t^*U_tdt$ with identical consequences. In this form the existence of $\Omega_\pm$ is an extension of von Neumann's mean ergodic theorem to simple scattering systems. It is also shown that U_t is unitarily equivalent to the reduction of V_t to the continuum states of H, implying that the continuum parts of the spectra of H and H_0 are identical. This latter relationship emerges here as a necessary condition for a system to be a scattering system. Physicists usually assume it to be also sufficient, though no proof to that effect seems to have been given. The paper concludes with a derivation of the scattering cross section from the scattering operator. This reviewer knows of no other such derivation which is rigorous and valid for relativistic as well as non-relativistic systems.

F. Rohrlich (Baltimore, Md.)

683:

Goldberg, Irwin. Transformation groups in a Q-number configuration space. Rev. Mod. Phys. **29** (1957), 450–451.

The quantum theoretical reformulation of classical theories frequently meets with difficulties that arise from ambiguities in the proper order of non-commuting factors, as well as from the cumbersome transformation

properties of canonical momentum components in theories that are to be relativistically invariant or gauge invariant. A number of authors have dealt with the possibility of circumventing some of these difficulties by basing the quantization procedure on the Lagrangian rather than the Hamiltonian version of a classical theory.

However, not all quantum effects, notably spin, are derivable from the quantization of a classical theory. In these circumstances, it is worthwhile to seek a method for determining the observables, their commutators and their equations of motion of a quantum theory based on a q-number Lagrangian. In this paper a program is discussed for extending a classical transformation group in configuration space to a q-number configuration space. (From the author's introduction.) *K. Yano* (Paris)

684:

Ciochină, Ioan N. Neue Versuche ueber die Anwendung der Wellenmechanik in der Chemie. Bul. Inst. Politech. Iaşi **4** (1949), 237–301.

685:

Datzeff, Assène. Sur le formalisme mathématique de la mécanique quantique. C. R. Acad. Sci. Paris **246** (1958), 1812–1815.

686:

Gasiorowicz, S.; and Ruderman, M. A. Identical scattering from causal and noncausal interactions. Phys. Rev. (2) **110** (1958), 261–264.

This paper deals with an illustrative example to show that one cannot tell from the observation of the scattering amplitude alone whether the interaction responsible for the scattering was causal or not. The reason for this fact is that the scattering amplitude specifies the wave function only far away from the scatterer and hence cannot yield unique information about what goes on in the region of interaction itself. Thus in general, an infinite number of interactions give the same scattering and hence the same dispersion relations. Some of these interactions might be causal, others might not. These statements are exhibited on the example of the scattering of a Klein-Gordon particle by a fixed static scalar interaction. The scattering amplitude is expressed in terms of the commutator of field operators far away from the scatterers only. Then a non-local, separable potential is constructed explicitly which gives precisely the same scattering at all angles and energies as the original potential. Finally it is shown that this second interaction is non-causal.

M. J. Moravcsik (Livermore, Calif.)

687:

Mead, C. A. Quantum theory of the refractive index. Phys. Rev. (2) **110** (1958), 359–369.

This is an elegant paper. An expression for the refractive index, defined as the ratio of vacuum and matter frequencies for a fixed wavelength, is derived by an approximate diagonalisation of the total Hamiltonian of the radiation field coupled to matter, using a method due to Bohm and Pines [same Rev. **82** (1951), 625–634; MR **12**, 886]. This agrees with the familiar semiclassical result. It is shown that the phenomenologic Maxwell equations in matter hold as operator equations for the quantized field, provided the state of the medium is considered fixed. As an application, the Frank-Tamm formula [C. R. Acad. Sci. SSSR **14** (1937), 109–114; J. Phy. (USSR) **1** (1939), 439–454] for Čerenkov radiation is obtained.

E. C. G. Sudarshan (Cambridge, Mass.)

688:

Omnès, R. On the solution of certain singular integral equations of quantum field theory. Nuovo Cimento (10) **8** (1958), 316–326.

The author considers the integral equation,

$$(*)\quad f(x)=g(x)+\frac{1}{\pi}\int_1^\infty \frac{h^*(x')f(x')}{x'-x-i\varepsilon}\,dx'+\int_1^\infty K(x,x')f(x')dx',$$

$1<x<\infty$, for $f(x)$, given g. $h(x)$ has the form $\exp(i\delta(x))\sin\delta(x)$ and K is regular. For the special case $K\equiv 0$ the equation is solved explicitly by function theoretic methods [compare the paper of this reviewer and A. E. Heins, Quart. J. Math. Oxford Ser. (2) **9** (1958), 132–143]. This explicit solution then permits the transformation of (*) to a regular Fredholm equation. Some applications are indicated in quantum theory. The role of the parameter ε is not clear to the reviewer and the method of solution, in fact, seems to be valid only for $\varepsilon=0$. *R. C. MacCamy* (Pittsburgh, Pa.)

689:

Scarf, F. L.; and Umezawa, H. Admissible solutions of the covariant two-body problem. Phys. Rev. (2) **109** (1958), 1848–1853.

Salpeter and Bethe [same Rev. **84** (1951), 1232–1242; MR **14**, 707] have proposed a covariant two-body equation which allows a completely relativistic approach to the bound state problems.

Wick [ibid. **96** (1954), 1124–1134, MR **16**, 655] and Cutkosky [ibid. **96** (1954), 1135–1141; MR **16**, 656] have been able to give an explicit solution to this equation in the case of a pair of scalar particles interacting through a neutral scalar field. Those solutions depend on four quantum numbers (n, m, l, k). Only for $K=0$ those states correspond to the well-known non-relativistic wave functions labelled by (n, m, l).

In this paper it is proved that the "spurions solutions" corresponding to $k>0$ cannot be considered as solution of the field theoretical problem. Indeed it is shown that the spurions solutions do not satisfy an asymptotic condition at $T\to-\infty$ that had been used by Gell-Mann and Low [ibid. **84** (1951), 350–351; MR **13**, 413] in order to deduce the homogeneous Bethe Salpeter equation from field theory. *S. Fubini* (Geneva)

690:

★**Umezawa, H. Quantum field theory.** Series in Physics. North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York; 1956. xv+364 pp. $9.75.

The author has tried, in this book, to cover the subject matter of quantum field theory in a manner both as general and as concise as possible, and what defects there are in his book can be traced directly to the adoption of such a laudable and unattainable goal.

After an excellent historical introduction, the classical field theories of particles of spin zero, one-half, and one are accorded a chapter each, followed by a chapter devoted to particles of arbitrary spin. From this point to the end of the book the particles of spin zero and one-half (which include all particles whose spin is known, except for photons) appear as examples of the general formalism. This has the consequence that the book will not be most useful to a student who wants a rudimentary grounding in field theory, but rather to an expert.

There is included an elegant description of the author's

own work on the theory of propagators, and the work included goes through to the beginning of 1956.

H. W. Lewis (Madison, Wis.)

691:

Spitzer, Richard. Commutation relations of interacting spinor fields. Phys. Rev. (2) **105** (1957), 1919–1923.

It is shown that the requirement that the Hamiltonian density commute with itself on a spacelike surface precludes the possibility that three or more spinor fields, coupled to one another in Yukawa-type interactions, commute with each other. If the Hamiltonian contains only two such fields, however, they may be assumed either to commute or to anticommute without violating this requirement. (Author's abstract.)

A. H. Taub (Urbana, Ill.)

692:

Halfin, L. A. Causality condition and criterion of physical realizability. Vestnik Leningrad. Univ. **12** (1957), no. 16, 5–18. (Russian. English summary)

"Causality conditions, the resulting dispersion relations and criteria of physical realizability in quantum field theory, classical electrodynamics, and the theory of electrical systems, are considered." (From the author's summary.)

I. E. Segal (Copenhagen)

693:

Ruijgrok, Th. W. Exactly renormalizable model in quantum field theory. I. The diagram analysis. Physica **24** (1958), 185–204.

This paper contains a study of a generalized Lee model introduced earlier by Van Hove and the author [Physica **22** (1956), 880–886]. The model contains a heavy nucleon with several discrete energy levels in interaction with a boson. The interaction contains a different coupling constant for each special state of the nucleon and is further governed by very special selection rules. These selection rules are constructed in such a way as to make an explicit construction of a few simple states in the model possible. In the paper reviewed here, the author makes very detailed perturbation theory computations of matrix elements of the S-matrix and of the renormalization constants. In particular, it is shown that the limit of a point interaction with equal values for all the renormalized coupling constants of the model is possible without the introduction of so called "ghost states". In this limit, there is no boson-nucleon scattering but a non-vanishing nucleon-nucleon interaction remains. {Reviewers note: In a paper by Dell'Antonio and Duimio [Nuovo Cimento **6** (1957), 751–754] it is shown that this result essentially depends on the condition that all renormalized coupling constants are put equal. If one tries to go to the point source limit with unequal values of these constants, one will, in general, find "ghost states" in the model.}

G. Källén (Lund)

694:

Ruijgrok, Th. W. Exactly renormalizable model in quantum field theory. II. The physical-particle representation. Physica **24** (1958), 205–213.

Continuing his earlier work (see the preceding review), the author reformulates his model of a renormalizable field theory in terms of what is called "asymptotically stationary states". These a.s. states are defined as direct products of physical one-particle states. Because of the interaction present in the model, these a.s. states are in general not eigenstates of the Hamiltonian, but the physical many-particle states can be expanded in terms of the a.s. states. The coefficients in this expansion satisfy a rather complicated system of coupled equations replacing the conventional Schrödinger equation. The author wants to study these equations because they are expressed in terms of renormalized quantities only.

G. Källén (Lund)

695:

Stehle, P. Calculation of electron-electron scattering. Phys. Rev. (2) **110** (1958), 1458–1460.

Matrix elements are given which permit the easy computation of Møller and Bhabha cross sections for all initial and final spin states.

F. Rohrlich (Baltimore, Md.)

696:

Rodberg, Leonard S. Energy shifts in the Feynman formalism. Phys. Rev. (2) **110** (1958), 277–278.

The relation between the S matrix and the energy-level shift is demonstrated in a form which permits the use of the Feynman methods of calculation. It is also shown that vacuum fluctuations and unlinked clusters do not contribute to the energy of a physical system. (Author's summary.)

F. Rohrlich (Baltimore, Md.)

697:

Elliott, J. P. Collective motion in the nuclear shell model. II. The introduction of intrinsic wave-functions. Proc. Roy. Soc. London. Ser. A. **245** (1958), 562–581.

In part I [same Proc., 128–145; MR **19**, 1136] the author formulated the problem of comparison of the collective and shell models of atomic nuclei. This paper extends the analysis. The one-particle wave functions of the shell model are taken to be of oscillator type. It has been shown [J. M. Jauch and E. L. Hill, Phys. Rev. **57** (1940), 641–645; MR **1**, 278] that in this case the symmetry group of the Schrödinger equation is isomorphic to the unitary unimodular group on 3 variables. When N particles are put in the same field the symmetry group is of the same type, but on $3N$ variables. Starting from this point, the author shows how to construct wave functions corresponding to different methods of factorization of this group, in such a manner as to lead to wave functions of the type employed in the collective model. The work is too involved in detail to describe here. Its success in a practical sense arises from the construction of a small set, called intrinsic wave functions, from which the remainder can be generated by rotation of the coordinate axes. This supplies the analogous step in quantum mechanical theory to the introduction of principal axes for the rotating atomic nucleus in classical mechanics. It is similar, in a mathematical sense, to the generation of spherical harmonics from the Legendre polynomials by rotations. {The apparent limitation to oscillator wave functions for the shell model could be lifted by the use of any other potential field for which the complete symmetry group of the Schrödinger equation, and the representation theory of the group, were known.}

E. L. Hill (Minneapolis, Minn.)

698:

Johnston, D. F. Space-group operations and time-reversal for a Dirac electron in a crystal field. Proc. Roy. Soc. London. Ser. A. **243** (1958), 546–554.

It is shown how the theory of spin-orbit coupling in the band theory of crystals can be developed directly from the Dirac equation without the usual ambiguities over improper rotations which arise in a treatment based on the Pauli-Schrödinger equation. It is shown that only the

double-value representations of the double point-groups are associated with solutions of the Dirac equation. A simple proof is given by Wigner's time reversal theorems for one-electron systems. *A. C. Hurley* (Melbourne)

699:

Brueckner, K. A.; Gammel, J. L.; and Weitzner, H. Theory of finite nuclei. Phys. Rev. (2) **110** (1958), 431–445.

The reaction-matrix theory of the nuclear many-body system is formulated for finite nuclei [for infinite nuclei, see K. A. Brueckner and J. L. Gammel, same Rev. **109** (1958), 1023–1039]. The determination of the reaction-matrix involves the consideration of coupled Hartree-Fock and reaction-matrix self-consistency problems and the paper is concerned with an approximate procedure for obtaining their solution. The approximation consists of formally replacing the reaction-matrix for the finite nucleus by the reaction-matrix for the local density, which is a nonlocal operator in coordinate space. Methods of dealing with the Hartree-Fock problem taking into account the non-local character of the reaction matrix are described. *A. Dalgarno* (Belfast)

700:

Takabayasi, Takehiko. The vector representation of spinning particle in the quantum theory. I. Progr. Theoret. Phys. **14** (1955), 283–302.

The author derives a set of vector equations equivalent to the two component Pauli equation for the spinning electron. These equations are interpreted in terms of a classical fluid with spin. The method of derivation involves writing the Lagrangian from which the Pauli equation may be obtained in terms of a basic set of scalars and vectors equivalent to the two component spinor. This method has the advantage of leading immediately to the stress-energy tensor of the "fluid". *A. H. Taub* (Urbana, Ill.)

701:

Szépfalusy, P. On the Fermi zero-point kinetic energy of particles with spin 1/2. Acta Phys. Acad. Sci. Hungar. **7** (1957), 433–446. (Russian summary)

In the usual calculation of the energy of a system consisting of particles with spin $\frac{1}{2}$ a summation over quantum states is replaced by an integration. In this paper the author presents an approximation process whereby the summands are first simplified and the summation is then carried out. Tables comparing the usual procedure with that proposed here are given. *A. H. Taub* (Urbana, Ill.)

702:

Matthews, P. T.; and Salam, Abdus. K-meson dispersion relations. I. Theory. Phys. Rev (2) **110** (1958), 565–568.

This is a discussion of some theoretical points connected with the application of the conjectured dispersion relations for K-meson scattering on nucleons. Our ignorance about the "unphysical continuum" (where the energy is enough for $\Lambda+\pi$ but not for $N+K$) is stressed. *J. C. Taylor* (London)

703:

Matthews, P. T.; and Salam, Abdus. K-meson dispersion relations. II. Applications. Phys. Rev. (2) **110** (1958), 569–572.

Present experimental data on K-p scattering is inserted into the dispersion relations, making plausible extrapolations (which have proved consistent with subsequent data) where necessary. The parities of Λ and Σ are assumed to be the same, the K^+-p potential is taken as repulsive, and the behaviour in the unphysical region is supposed not too wild. The tentative conclusion is that the K-meson is scalar or pseudo-scalar depending on whether the K^--p potential turns out repulsive or attractive. *J. C. Taylor* (London)

704:

Rzewuski, J. On inversions of space and time. Acta Phys. Polon. **16** (1957), 435–445. (Russian summary)

The transformation properties of two-component spinors with respect to inversions of space and time are discussed in some detail. The two component theory of the neutrino proposed by Landau, Salam, and Lee and Yang is compared to a Majorana type of theory. A geometrical discussion of the theorem of Lüders and Pauli is given. *A. H. Taub* (Urbana, Ill.)

705:

Finkelstein, R. General covariance and elementary particles. Phys. Rev. (2) **110** (1958), 1200–1203.

This is an exploratory paper in which the author tries to combine the concept of quantization of covariant field theories with that in flat spaces to yield a description of elementary particles. The basic idea seems to be that of considering 4-dimensional curved space-time as a subspace of a larger euclidean, or pseudo-euclidean, flat space. In this flat space, one can make use of spinor analysis in the usual form, based on linear transformations, for the description of elementary particles. {While the author does not arrive at any very specific proposals on the side of physical theory, the paper is interesting mathematically and is suggestive for further development.} *E. L. Hill* (Minneapolis, Minn.)

706:

Hammer, C. L.; and Good, R. H., Jr. Quantization process for massless particles. Phys. Rev. (2) **111** (1958), 342–345.

This paper gives the quantization of a recently proposed theory for particles of arbitrary spin and zero mass. An interesting result is that there is a connection between the spin and the statistics of the particles. It is found that the spinor components of a boson/fermion field with integral/half-integral spin commute/anticommute off the light cone whereas the spinor components of a boson/fermion field with half-integral/integral spin do not. As in the unquantized theory, the two-component neutrino and the photon are special cases. (Authors' abstract) *C. N. Yang* (Princeton, N.J.)

707:

Hammer, C. L.; and Good, R. H., Jr. Wave equation for a massless particle with arbitrary spin. Phys. Rev. (2) **108** (1957), 882–886.

A wave equation for a noninteracting particle with zero mass and arbitrary spin s is given in this paper. The Hamiltonian is proportional to the inner product of the momentum and spin operators so that the wave function has $2s+1$ components. As an auxiliary condition, solutions with spins not parallel or anti-parallel to the momentum are discarded. With this condition the theory is Lorentz-covariant. The energy, momentum, and angular momentum are defined in terms of expected values of the usual type of displacement operator. The specialization $s=\frac{1}{2}$ is the two-component neutrino theory and $s=1$ gives Maxwell's equations for the photon. (Author's abstract.) *C. N. Yang* (Princeton, N.J.)

708:
Ito, D.; and Tanaka, H. Theory of meson multiple production. Nuovo Cimento (10) **7** (1958), supplemento, 91–116.

A theory of multiple meson production is described which incorporates into a quasi-field-theoretic framework the perfect gas model of Fermi, and the hydrodynamic model of Landau. The theory gives reasonable agreement with experiment. *H. W. Lewis* (Madison, Wis.)

709:
Foldy, Leslie L. Synthesis of covariant particle equations. Phys. Rev. (2) **102** (1956), 568–581.

The author shows that by combining two irreducible representations of the proper inhomogeneous Lorentz group certain irreducible unitary representations are obtained of the extended Lorentz group which includes space and time inversions. The author gives the Schrödinger equation whose solutions constitute the representation space for the above representations. It is shown that the Dirac, Klein-Gordon and Proca equations give representations of the type described above. It is further shown that other possible equations exist for particles of spin zero, one-half and one as well as for arbitrary spin. These possibilities occur because the operators corresponding to space inversion, time inversion and charge conjugation are not uniquely determined by the invariance requirements and the method of construction of the representation space. In fact, two possibilities for each operator exist. It is conjectured that second quantization will lead to different theories when different choices are made for these operators. *A. H. Taub* (Urbana, Ill.)

710:
Tzou, Kuo-Hsien. Sur une théorie non linéaire des particules de spin. I. C. R. Acad. Sci. Paris **246** (1958), 1815–1817.

711:
Macke, Wilhelm. Das quantenmechanische Vielteilchenproblem bei Atomkernen und Elektronengasen. Ann. Physik (6) **20** (1957), 80–92.

A systematic study is made of the Brueckner-Bethe theory for atomic nuclei which takes into consideration the correlation between two particles; the point of view is that of a perturbation calculation of all orders. Some simplifications are suggested, with the result that certain elements in the scatter matrix are simplified. Then the degrees of freedom still available in the definition of an average potential are used to remove terms of order $1/A$ from the energy. At the same time a method is developed for obtaining the higher order interactions. The contributions neglected in these higher order terms are due to multi-scatter alone. *C. Froese* (Vancouver, B.C.)

712:
Galitskii, V. M.; and Migdal, A. B. Application of quantum field theory methods to the many body problem. Soviet Physics. JETP **34(7)** (1958), 96–104.

This paper lays the groundwork for an application of field-theoretic techniques to the problem of an interacting system of particles obeying Fermi statistics. The single-particle Green's function is described, and its structure is related to the properties of "quasi-particles". The two-particle Green's function is discussed, in the case of repulsive forces between particles. Applications of the formalism are postponed to a later publication.
H. W. Lewis (Madison, Wis.)

RELATIVITY

See also 465, 466, 614, 707, 758.

713:
Marx, György; and Román, Pál. Energy and momentum in the general theory of fields. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 269–287. (Hungarian)

This paper contains an exposition of the different methods one can use to generate the energy-momentum tensor in field theories. These methods are based on the invariance of the Lagrangian under certain variations. 1) Hilbert's method: The invariance of the Lagrangian with respect to variations of the metric tensor gives the (symmetric) energy-momentum tensor; 2) Belinfante's method: The invariance of the Lagrangian with respect to variations which correspond to infinitesimal inhomogeneous Lorentz transformations enables us to construct a nonsymmetric canonical energy-momentum tensor which can be symmetrized, giving the energy-momentum tensor. The authors then show that the two methods generate the same energy-momentum tensor, if the system is closed (i.e., if the divergence of the energy-momentum tensor is zero); if the system is open, only method 1) can be used. Large numbers of applications are given; the authors construct the energy-momentum tensor for scalar fields, pseudo-scalar fields, spinor fields and Maxwell-fields, without and with sources.
N. L. Balazs (Chicago, Ill.)

714:
Unal, Burhan Cahit; et Vigier, Jean-Pierre. Introduction des paramètres relativistes d'Einstein-Kramers et de Cayley-Klein dans l'hydrodynamique relativiste du fluide à spin de Weyssenhoff. C. R. Acad. Sci. Paris **245** (1957), 1890–1892.

The authors give a Lagrangian function which leads to the equations of motion proposed by Weyssenhoff for a classical fluid with spin. The Lagrangian is written in terms of an orthogonal enuple and in terms of bilinear forms of a four-component spinor field.
A. H. Taub (Urbana, Ill.)

715:
Stiegler, K. L'hypothèse d'ondes corpuscules et la théorie de la relativité restreinte. Nuovo Cimento (10) **8** (1958), 922–926.

The Lorentz transformation is deduced from a set of axioms which differs from the usual by replacing the assumption of the constancy of the velocity of light by the assumption of the existence of a wave corresponding to a particle. *C. W. Kilmister* (London)

716:
Komjáthy, Aladár. A new, simple derivation of the Lorentz transformations. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 179–182. (Hungarian)

The paper claims to give a new and simple derivation of the Lorentz transformations and claims to "explain" why $(1-\beta^2)$ appears on the $\frac{1}{2}$ power. In the usual derivations [e.g., M. Abraham and R. Becker, Theorie der Elektrizität, Leipzig 1933, Vol. II, pp. 269–272] there is no ambiguity concerning the exponent of $(1-\beta^2)$. Furthermore, the novelty of the present derivation consists mainly in a slight rearrangement of the observers, measuring rods, and light sources in the various coordinate systems. Thus, the value of the paper is, if anything, didactic. *M. J. Moravcsik* (Livermore, Calif.)

717:

Shibata, Takashi. Improved theory of the new fundamental group of transformations in special relativity. Mem. Fac. Engrg. Hiroshima Univ. **1** (1957), 1–8.

The author deals with the sub-group of the general Lorentz group, which leaves invariant the equation of plane wave front advancing in a fixed direction with the velocity of light. The so-called special Lorentz transformations, which represent the relations between the space-time coordinates of two observers moving relatively to each other with uniform velocity, are the most important transformations since all the new ideas of the special theory of relativity are based upon the special Lorentz transformations. — In general, the combination of two "Lorentz transformations without rotation" [cf. C. Møller, The theory of relativity, Clarendon, Oxford, 1952; MR **14**, 212] is not of the same form but is equivalent to a Lorentz transformation with a rotation. The problem is to modify the transformations

$$x'=\frac{x-ut}{\sqrt{(1-\beta^2)}},\quad y'=y,\quad z'=z,\quad t'=\frac{t-ux/c^2}{\sqrt{(1-\beta^2)}},$$

$$\beta=\frac{u}{c},\quad u^2=(u^1)^2+(u^2)^2+(u^3)^2,$$

so that the modified transformations, regarding u^1, u^2, u^3 as parameters, form a 3-parameter group (Postulate of group property: the transformations of the space-time coordinates in two systems moving relative to each other with uniform velocity u^h, regarding the three components of the velocity as parameters, should form a 3-parameter group). The problem can be solved by abandoning the assumption that the components of the position vector x^h in the direction perpendicular to the velocity u^h undergo no change and by using the assumption of the invariance of the direction of the ray of light which propagates in a fixed direction. An equivalent expression is the assumption of invariance of the equation of a plane wave front advancing in a fixed direction with the velocity of light. *M. Pinl* (Cologne)

718:

Kalitzin, N. S. Grundgleichungen der relativistischen Mechanik eines materiellen Punktes mit veränderlicher Masse. Nuovo Cimento (10) **8** (1958), 843–849.

Verfasser verallgemeinert die zuerst von W. Meschtcherski angegebene nichtrelativistische Grundgleichung der Mechanik eines materiellen Punktes mit veränderlicher Masse auf den Bereich relativistischer Geschwindigkeiten. Im Bereich klassischer Geschwindigkeiten vom Betrag v hatte sich die aus der Meschtcherskischen Gleichung die klassische Formel $v=q\lg(m_0/m)$ ergeben, auf welcher Grund welcher die Bewegung ein- und mehrstufiger Raketen berechnet wird (dabei entspricht m_0 der Anfangsmasse für $v=v_0$, q ist als Geschwindigkeitsbetrag der weggeschleuderten Teilchen in Bezug auf die Rakete zu deuten). Wenn die zu v und q gehörenden Vektoren linear abhängig sind, gilt

$$\frac{m_0}{m}=\left(\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}\right)^{c/2q}$$

wie auch schon von E. Sänger angegeben worden ist. Als Spezialfall wird auch die "reine" Rakete behandelt, die sich in einem von Gravitation, Energie und Materie freien Raum bewegt. *M. Pinl* (Köln)

719:

Makarova, N. M. On the geometry of Galilei-Newton. Oreh.-Zuev. Ped. Inst. Uč. Zap. **1** (1955), 83–95. (Russian)

720:

O'Raifeartaigh, L.; and Synge, J. L. A property of empty space-time. Proc. Roy. Soc. London. Ser. A. **246** (1958), 299–300.

The following theorem is proved: If the Ricci tensor $R_{mn}=0$ throughout a V_4, and the curvature tensor $R_{rmns}=0$ on a smooth non-null 3-space in V_4, then $R_{rmns}=0$ in the neighbourhood of V_4, and also in general throughout the whole of V_4.

C. Gilbert (Newcastle-upon-Tyne)

721:

O'Raifeartaigh, L. A static generalization of the Einstein universe. Proc. Roy. Soc. London. Ser. A. **245** (1958), 202–212.

The author constructs static, spherically symmetric solutions of Einstein's gravitational field equations (with cosmical term) representing spheres of matter of constant density μ surrounded by matter of different constant density ν. He finds inter alia that the space can be simply connected and open only if $\nu=0$.

F. A. E. Pirani (Chapel Hill, N.C.)

722:

Mizkjewitsch, Nikolaj. Zu den Invarianzeigenschaften der Lagrange-Funktionen der Felder. Ann. Physik (7) **1** (1958), 319–333.

It is well known that one may obtain detailed information concerning the conservation laws of physical quantities from the invariance of the action integrals. An equivalent method is developed, this method depending solely on the invariance properties of the Lagrangian function. This enables the author to obtain "differential" conservation laws which do not involve the use of integrals. A significant feature of this method is the fact that it automatically gives rise to a generalisation of the notion of angular momentum applicable in the general theory of relativity. The quantities which — according to the author's theory — are conserved are classified and discussed under the headings of energy and angular momentum, giving rise to expressions which are interpreted as multi-moments and spin quantities. In order to devise a method according to which the actual form of the Lagrangian function itself should be determined, the author postulates a number of conditions which should govern such a choice, these being suggested on the grounds of formal simplicity ("Einfachheitsprinzip"). A new type of unitary theory for fields with quanta of vanishing rest-mass is formulated on the basis of such principles.

H. Rund (Durban)

723:

Anderson, James L. Reduction of primary constraints in generally covariant field theories. Phys. Rev. (2) **111** (1958), 965–966.

Almost simultaneously with P. A. M. Dirac [# 724, 725 below] and with B. S. De Witt [unpublished lecture], the author has succeeded in removing the redundance of the canonical variables of general relativity partially by means of a canonical transformation affecting four of the canonical momentum densities. Whereas the number of non-redundant canonical field variables is believed to be four per three-dimensional space point, and the number of field variables prior to reduction is twenty per space point, they are reduced to twelve per space point by the present paper. *P. G. Bergmann* (Syracuse, N.Y.)

724:

Dirac, P. A. M. Generalized Hamiltonian dynamics. Proc. Roy. Soc. London. Ser. A. **246** (1958), 326–332.

In relativistic field theories, the momenta $p_n=\partial L/\partial \dot{q}_n$ are not independent functions of the velocities $\dot{q}_n$. The author devised his general dynamical theory [Canadian J. Math. **2** (1950), 129–148; MR **13**, 306] to deal with this situation. In the present paper he reviews the theory and shows how under certain conditions the number of degrees of freedom of the dynamical system may be reduced. Sufficient conditions are: (1) that some of the primary constraints (first class φ's) be linear in the momentum variables with constant coefficients; (2) that none of the secondary constraints (χ equations) be independent of the momentum variables. The method of reduction is to add a suitable total derivative to the Lagrangian.

The author remarks that his method is more general than the alternative method of Anderson and Bergmann [Phys. Rev. (2) **83** (1951), 1018–1025; MR **13**, 411].

F. A. E. Pirani (Chapel Hill, N.C.)

725:

Dirac, P. A. M. The theory of gravitation in Hamiltonian form. Proc. Roy. Soc. London. Ser. A. **246** (1958), 333–343.

The methods of the paper reviewed above are used to put Einstein's theory of gravitation into Hamiltonian form. The author goes beyond previous work [e.g., P. G. Bergmann, R. Penfield, R. Schiller, and H. Zatzkis, Phys. Rev. (2) **80** (1950), 81–88; MR **12**, 292; F. A. E. Pirani, A. Schild, and R. Skinner, same Rev. (2) **87** (1952), 452–454; MR **14**, 418] by eliminating from the theory the degrees of freedom corresponding to the field variables (metric tensor components) $g_{\mu 0}$ ($\mu=0, 1, 2, 3$) [this part of the work parallels independent work of J. L. Anderson, # 723 above].

Using a linear approximation, he concludes that the gravitational radiation field has two degrees of freedom and positive definite energy density. He concludes also that the gravitational force between positive masses must be attractive.

{The reduction in the number of degrees of freedom is achieved at the cost of manifest Lorentz invariance. The author is "inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world." The reviewer finds it difficult to concur.}

F. A. E. Pirani (Chapel Hill, N.C.)

726:

Lichnerowicz, André. Sur un procédé de quantification du champ de gravitation. C. R. Acad. Sci. Paris **247** (1958), 433–436.

1. Following his earlier definition of pure gravitational radiation, [same C. R. **246** (1958), 893–896; MR **19**,237], the author replaces the usual Einstein equations by the field equations

$$S_{[\alpha\beta\gamma]}R_{[\alpha\beta|\lambda\mu;|\gamma]}=0, \quad R^{\alpha}{}_{\beta\lambda\mu;\alpha}=0,$$

the symbol S denoting cyclic permutation, with the Einstein equations $R_{\alpha\beta}=0$ being satisfied on an initial hypersurface. Both the Einstein equations and the second set of new equations are then satisfied everywhere, whereas the first set represents the Bianchi identities. A metric field satisfying these conditions might be considered the gravitational analogue of a normal mode in a Lorentz-covariant linear field theory.

2. With the linearized theory, the author carries out a process of quantization corresponding to the usual procedures. In view of the fact that the rigorous theory does not obey the principle of linear superposition, it is not clear at present whether Lichnerowicz, approach is capable of being extended to it.

P. G. Bergmann (New York, N.Y.)

727:

Sciama, D. W. On a non-symmetric theory of the pure gravitational field. Proc. Cambridge Philos. Soc. **54** (1958), 72–80.

The energy momentum tensor of a material system whose field possesses intrinsic spin is non-symmetric. The author considers a pure theory of gravitation in which such a material field is to be phenomenologically accommodated. The mathematical apparatus of Einstein's theory involving an asymmetric fundamental tensor is therefore taken over, for the metrical energy momentum tensor (i.e. the hamiltonian derivative of the Lagrangian of the field) is asymmetrical in such a theory. This procedure is not intended to have anything to do with unified field theory. Conservation laws, angular momentum, and the motion of test particles are treated in some detail.

H. A. Buchdahl (Princeton, N.J.)

728:

Anderson, James L. Enumeration of the true observables in gauge-invariant theories. Phys. Rev. (2) **110** (1958), 1197–1199.

The paper deals with a theory in which the field equations are invariant under a gauge group. It is shown that in general the number of "true" variables, i.e., those whose motion is uniquely determined, is given by $N-2n$, where N is the number of field variables and n is the number of arbitrary functions needed to specify an element of the gauge group.

N. Rosen (Haifa)

729:

***Ginzburg, V. L. The use of artificial earth satellites for verifying the general relativity theory.** The Russian literature of satellites. I, pp. 175–181. Translated from Uspehi Fiz. Nauk **63** (1957), no. 1a. International Physical Index, Inc., New York, 1958. vii+181 pp. (1 plate) $10.00.

The paper summarizes the present position with regard to the subject of the title.

C. Gilbert (Newcastle-upon-Tyne)

730:

Vaidya, P. C.; and Shah, K. B. A radiating mass particle in an expanding universe. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 534–539.

A spherical mass particle, embedded in an expanding universe, is considered. The particle is emitting radiation. The metric of the space-time external to the particle has spherical symmetry about the particle which is located at the spatial origin. A time-coordinate t is employed such that t=constant is the equation of a null-sphere with center at the origin. The coordinate-system is shown to be non-orthogonal. Einstein's equations for the metric are solved by approximation up to the second order. The solution obtained is a generalization of one found by McVittie [Monthly Not. Roy. Astr. Soc. **93** (1933), 325–339].

G. C. McVittie (Urbana, Ill.)

731:

Klein, O. The Dirac theory of the electron in general relativity theory. Norske Vid. Selsk. Forh., Trondheim **31** (1958), no. 8, 5 pp.

The Dirac equation for a massless spinor field $\Psi(x)$ may be written in terms of general coordinates x^μ as

$$\gamma^\mu(\partial_\mu-\Gamma_\mu)\Psi=0,$$

where the γ^μ are 4×4 matrices related to the metric tensor $g_{\mu\nu}$ by the usual anticommutation rules, $\{\gamma^\mu, \gamma^\nu\}=2g^{\mu\nu}$, and the Γ_μ are the matrices which must be introduced to define covariant differentiation of a spinor, i.e., the spinor affine connexion. Starting from the invariance under general coordinate transformations of the action integral associated with this equation of motion, the author derives the transformation properties of γ^μ and Γ_μ.

P. W. Higgs (London)

732:

Møller, C. On the localization of the energy of a physical system in the general theory of relativity. Ann. Physics 4 (1958), 347–371.

The author seeks a new definition of the total energy-density (matter plus gravitational) of a system satisfying Einstein's field equations. His main concern is to find a definition which has physical meaning for arbitrary transformations of the spatial co-ordinates. After surveying the difficulties associated with Einstein's definition

$$\theta_i^j=T_i^j+t_i^j,$$

where T_i^j is the matter-tensor and t_i^j is the canonical pseudo-tensor of the gravitational field, he proposes the alternative definition

$$I_i^j=\frac{\partial\chi_i^{jl}}{\partial x^l},$$

where

$$\chi_i^{jl}=\frac{\sqrt{(-g)}}{\kappa}\left(\frac{\partial g_{in}}{\partial x^m}-\frac{\partial g_{im}}{\partial x^n}\right)g^{jm}g^{ln}.$$

This new quantity I_i^j has all the desirable properties of θ_i^j, but, in addition, I_4^j transforms like a 4-vector density under arbitrary transformations of the spatial co-ordinates. Thus, for instance, one obtains physically sensible results if one uses "spherical polars", which is not the case with Einstein's definition.

A further property of the new energy-density (but not the momentum-density) is that it has a well-defined meaning for a non-closed system. Moreover, the total energy of a spatially closed universe now has an unambiguous meaning, and is in fact zero. The total matter energy in such a universe may be changing with time, but it is always exactly counterbalanced by the negative gravitational energy.

It is noteworthy that the gravitational part of I_i^j, that is, $I_i^j-T_i^j$, depends on the Riemann curvature tensor (in contrast to t_i^j), so that it should be useful in problems concerning gravitational radiation. The author promises to discuss this point in another paper.

D. W. Sciama (London)

733:

Cahen, Michel. Trajectoires de Schwarzschild et trajectoires de Newton. C. R. Acad. Sci. Paris **246** (1958), 386–388.

The trajectories of Kepler define an affine connection and thus determine a space K_4. The author shows that there is no Schwarzschild space S_4 in geodesic correspondence with K_4. *A. Raychaudhuri* (Calcutta)

734:

Synge, J. L. A model in general relativity for the instantaneous transformation of a massive particle into radiation. Proc. Roy. Irish Acad. Sect. A. **59** (1957), 1–13.

In a spherically symmetric universe, the author introduces a coordinate system (x^1, θ, ϕ, x^4) where the x^1, x^4 lines are null geodesics and θ, ϕ are angle coordinates. In this coordinate system, Einstein's field equations are solved for empty space. The fitting of the "interior" and "exterior" solutions separated by a thin shell of radiation (radiation density $\sigma\to0$ for $x^1\neq0$ and $\sigma\to\infty$ for $x^1=0$ with $\int\sigma dx^1=\Omega$, a finite function of x^4) is then discussed. The author opines that the solution corresponds to a mathematical point mass being instantaneously converted into a shell of radiation and may be of interest in the discussion of the rapid conversion of a star into radiation or of fundamental particles into radiation.

A. Raychaudhuri (Calcutta)

735:

Lameau, Jean. Solution à symétrie sphérique des équations de la relativité générale, en choisissant, comme tenseur d'impulsion-énergie, le tenseur de la théorie électromagnétique de Born-Infeld. C. R. Acad. Sci. Paris **245** (1957), 2208–2210.

The title expresses the contents of this paper. The results are that if in the field equations the cosmological constant $\Lambda=0$ and the gravitational constant equals one (in certain units of length), then

$$ds^2=-e^\mu dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)+e^{-\mu}dt^2$$

where

$$re^{-\mu}=r+\frac{r^3}{3}-\int_0^r\sqrt{(1+r^4)}dr.$$

L. Infeld (Warsaw)

736:

Born, M.; und Biem, W. Zum Uhrenparadoxon. Nederl. Akad. Wetensch. Proc. Ser. B. **61** (1958), 110–120.

The authors stress the difference between inertial and non-inertial systems. They argue that a complete discussion of the 'clock-paradox' should be made in terms of general relativity, because accelerations are involved. They think, however, that the discussion given by C. Møller in his book "The theory of relativity" [Oxford, 1952; MR **14**, 212; pp. 258–263] is too complicated and needs clarification. They therefore give a careful alternative analysis in which the (x, t) coordinates of the clock which undergoes acceleration are expressed as functions of proper-time. They discuss the general case (and a particular example) of relative motion with the two clocks coincident at the beginning and end, calculate the time-lag of the accelerated clock and explain why there is no paradox. *G. J. Whitrow* (London)

737:

Das, Anadijiban. Effects of the central spin on planetary motion. Progr. Theoret. Phys. **17** (1957), 373–382.

Lense and Thirring [Phys. Z. **19** (1918), 156–163] calculated (i) the gravitational field due to a homogeneous, uniformly rotating sphere assuming that Einstein's linearized law of gravitation holds, and (ii) the secular variations in the six Newtonian orbital elements using the standard methods of the perturbation theory.

The author extends these calculations up to second degree terms in the angular velocity ω of a rotating spheroid, making use of the external gravitational potentials as given by Clark [Proc. Cambridge Philos. Soc. **43** (1947), 164–177; MR **8**, 496]. He also investigates the motion of spinning test-particles, using the equations obtained by Papapetrou, and Corinaldesi and Papapetrou [Proc. Roy. Soc. London. Ser. A **209** (1951), 248–258, 259–268; MR **13**, 695].

Rayner [ibid. **232** (1955), 135–148; MR **17**, 545] calculated the gravitational field due to a rotating sphere

using Whitehead's law of gravitation. In the present paper, the author shows that Einstein's and Whitehead's theories differ in terms containing ω^2. Finally, he calculates the deflection of light rays and the perturbing forces acting on the six Newtonian orbital elements.

E. Leimanis (Vancouver, B.C.)

738:

Lameau, Jean. Discussion d'une solution particulière des équations de la relativité générale, obtenue en choisissant comme tenseur d'impulsion-énergie, le tenseur de la théorie électromagnétique de Born-Infeld. C. R. Acad. Sci. Paris **246** (1958), 1384–1386.

L'A. discute la solution particulière:

$$r\gamma = r + (\Lambda+\chi)\frac{r^3}{3} - \chi r_0^2 \int_\alpha^r \sqrt{\left(1+\frac{r^4}{r_0^4}\right)}dr$$

[voir l'analyse # 735 ci-dessus]. γ est infini d'ordre $1/r$ pour r tendant vers l'infini, et ne reste borné pour $r=0$ que si $\alpha=0$. L'A. étudie ensuite le tenseur d'impulsion énergie et montre que pour $r=0$, $T_1{}^1$ et $T_4{}^4$ sont infinis.

Y. Fourès-Bruhat (Marseille)

739:

Takasu, Tsurusaburo. Die endgültige kugelgeometrische Relativitätstheorie, welche als eine Faserbündelgeometrie aufgefasst ist. Yokohama Math. J. **4** (1956), 119–146.

In previous publications [same J. **1** (1953), 89–104, 263–273; MR **15**, 358; **16**, 184] the author established a correspondence between the formalism of the general theory of relativity and "non-holonomic" Laguerre geometry of the second kind, the development of the latter also being due to the author [ibid. **1** (1953), 1–87; MR **15**, 350]. The present paper is concerned with further developments in this direction. In the introduction the following results are announced: (1) The general theory of relativity does not yield true equations of motion, since the momentum vector of a particle is not parallel to the tangent of the geodesic representing its path; (2) The generalised gravitational theory of Einstein of 1953 contains a contradiction; (3) The final, universal, natural and essentially unitary field theory is the author's non-holonomic Laguerre fibre-bundle geometry of the second kind. — {Due to lack of clarity as regards terminology, notation and diction, as well as continual reference to 13 not easily accessible earlier memoirs, the reviewer was quite unable at any stage to follow the author's reasoning.}

H. Rund (Durban)

740:

Valcovici, Victor. Sur les principes fondamentaux de la théorie de la relativité. Rev. Math. Pures Appl. **2** (1957), 323–325.

741:

Mishra, R. S. A study of Einstein's equations of unified field. Nuovo Cimento (10) **8** (1958), 632–642.

L'auteur étudie les équations d'Einstein déduites d'un principe variationnel [Einstein, The meaning of relativity. 5e éd., Princeton Univ. Press, 1955; MR **17**, 907; app. II] $N^{ik}{}_{,l}=0$; $S_{ik}=0$. Il montre la forme des solutions en Γ^l_{jk} si elles existent, et celle de la solution approchée au 1er ordre des g_{ik} pour un champ faible. Ces solutions sont les mêmes que celles déduites du 1er système des équations d'Einstein [même livre 4e éd. 1953; MR **14**, 805; **15**, 357; app. II].

Il forme ensuite un autre ensemble d'équations du champ en remplaçant S_{ik} par son transposé $\tilde{S}_{ik}=S_{ki}$ dans l'hamiltonien, et calcule pour ce nouveau système d'équations la forme de la solution en Γ si elle existe. Pour ce système d'équations du champ il déduit des identités de Bianchi, en utilisant l'invariance de l'hamiltonien par rapport à un changement de coordonnées infinitésimal, et par rapport à une $\tilde{\lambda}$-transformation.

L'étude de ces équations (solution sous forme tensorielle) sera poursuivie dans de prochains articles.

J. Renaudie (Rennes)

742:

Flint, H. T.; and Williamson, E. M. The theory of relativity, the electromagnetic theory and the quantum theory. Nuovo Cimento (10) **8** (1958), 680–698.

This paper describes a five-dimensional theory in which particles acted on by gravitational and electromagnetic forces move along null geodesics. The fifth dimension is thereby related to the proper-time of the particles. A theory of measurement is then proposed, in which a gauging factor is identified with a spinor satisfying the Dirac equation. A minimum length is introduced which is related to the mass of the electron. This mass is determined partly by geometrical quantities and partly by interactions which enter naturally into the theory. It is claimed that this theory establishes a unity between gravitational, electromagnetic and quantum phenomena.

D. W. Sciama (London)

743:

Pachner, Jaroslav. Ergänzung zum Variationsprinzip für klassische Feldtheorien. Ann. Physik (7) **1** (1958), 201–202.

In a recent article [Ann. Physik (6) **19** (1956), 353–368; MR **19**, 615] the author discussed in detail the mathematical and physical aspects of a variational principle with reference to a unitary field theory. In an addendum, the functional dependence of the action of particles on the parameters of the latter is discussed, and the author concludes that this dependence should be determined by the solutions of the field equations. *H. Rund* (Durban)

744:

Sciama, D. W. On a geometrical theory of the electromagnetic field. Nuovo Cimento (10) **8** (1958), 417–431.

Weyl's gauge theory is generalized by allowing complex vierbeins (and therefore, for instance, a complex Hermitian metric). Any matter field is supposed to transform as a density of a certain weight (the charge) under the quasi-unitary group associated with the vierbeins. A theory of electromagnetism results, the potential being identified with a contraction of the antisymmetric part of the (Hermitian) connection. Tensor as well as spinor fields can carry charge. The correspondence principle leads to quantization of charge for bosons. Parity conservation for the electromagnetic interactions is guaranteed.

J. C. Taylor (London)

745:

Tiwari, Ramji. A particular solution of the field equations in Einstein's generalized theory of gravitation. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 293–305.

An approximate solution of Einstein's Unified Field Theory [The meaning of relativity, 4th ed., Princeton, 1953; MR **14**, 805] with the following properties is found. The solution reduces to the Schwarzschild solution in the absence of electromagnetic field and to the field of uniform monochromatic radiation, up to first order terms, in the absence of mass. Various other properties of the solution are given. *M. Wyman* (Edmonton, Alta.)

746:

Teodorescu, Ion. Le cas du champ électrogravifique symétrique sphérique et statique dans la théorie unitaire non holonome. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 9, 9–24. (Romanian. Russian and French summaries)

747:

Szekeres, G. Spinor geometry and general field theory. J. Math. Mech. **6** (1957), 471–517.

"The purpose of the present work is to exploit, more fully than has been done heretofore, the possibilities of spin connection from the point of view of geometrical field theories and to develop a geometry whose connection is derived exclusively from the displacement of spinors···. The external fields are described by the metric tensor and a complete set of skew tensors of orders 0, 1, 2, 3, 4···; there is no ambiguity in identifying the electromagnetic vector potential A_μ··· [If] L^μ is the current vector, then the wave equation is not $\Box A^\mu = L^\mu$ but $\Box\Box A^\mu = L^\mu$;··· there is a nowhere vanishing scalar field A whose inverse is interpreted as cosmic time··· The particular form of the Lagrangian requires the existence of an absolute unit of length···" (From the author's introduction) *F. A. E. Pirani* (Chapel Hill, N.C.)

748:

Dicke, R. H. Gravitation without a principle of equivalence. Rev. Mod. Phys. **29** (1957), 363–376.

A theory of gravitation is formulated in a flat space-time. Gravitation is considered to be associated with the variation in the electric polarizability of space and is therefore described by means of a scalar dielectric "constant" ε which is a function of position and time. Among various other assumptions, it is assumed that the magnetic permeability μ is equal to ε and therefore that the velocity of light c is inversely proportional to the latter. Starting with certain Lagrangian functions, equations of motion are set up for a particle in a given ε field and also field equations are obtained to determine ε. From these and the assumed relation between light frequency and ε, the "crucial tests" of the general relativity theory are accounted for. Also, the Maxwell equations for the electromagnetic field are obtained from a more general Lagrangian function. Some cosmological considerations are given. *N. Rosen* (Haifa)

ASTRONOMY

See also 609, 610, 613, 737.

749:

✱Okhotsimskii, D. E.; Eneev, T. M.; and Taratynova, G. P. Determining the lifetime of an artificial earth satellite and investigating the secular perturbations of its orbit. The Russian literature of satellites. I, pp. 45–70. Translated from Uspehi Fiz. Nauk **63** (1957), no. 1a. International Physical Index, Inc., New York, 1958. vi+181 pp. (1 plate) $10.00.

The authors consider a satellite on an orbit which is affected by air drag. The air drag is assumed to be proportional to the density of the air and to the square of air velocity and the density is taken as a given function of the altitude.

A first order perturbation analysis employing a small air drag parameter is used to reduce the problem of determining the lifetime of a satellite to the integration of a system of two first order differential equations. In the case of a central gravitational field the authors' analysis is basically equivalent to the analysis presented by R. E. Roberson [see #474 above].

Numerical solutions of the above system are worked out and given in tabular form for a particular type of density distribution. If the reviewer's arithmetic is correct the lifetimes determined from these tables considerably overestimate the observed lifetimes.

In the final section of the paper the effects of the oblateness of the earth and the diurnal rotation of the atmosphere are considered and it is shown that lifetimes computed by taking into account these effects differ only by 10–20% from the previously mentioned lifetimes.

E. T. Onat (Providence, R.I.)

750:

✱Iatsunskii, I. M. The effect of geophysical factors on the motion of a satellite. The Russian literature of satellites. I, pp. 86–106. Translated from Uspehi Fiz. Nauk **63** (1957), no. 1a. International Physical Index, Inc., New York, 1958. vi+181 pp. (1 plate) $10.00.

The effects upon the orbit of an artificial satellite of the following geophysical factors are studied:

a. Errors in the assumed mass of the earth, its oblateness, the ratio of centrifugal force to gravity at the equator, and the equatorial radius. It is found that the effects of a plausible error in the oblateness greatly exceeds the effects of plausible errors in the other quantities. The equations give only the first-order effects of the oblateness error.

b. Spherical harmonics in the gravitational field other than that represented by the oblateness. It is concluded on the basis of numerical integrations of particular cases that the effects will be large, especially on the quantity p, (the orbital parameter, incorrectly translated the "element"). Effects up to 1.2 km are predicted after five revolutions.

c. Air resistance. At the time the paper was written, the air drag could not be predicted within a factor 10. A study of the latitude variations of air resistance indicated that the effect was negligible.

Equations are set up for solving simultaneously for these unknowns from the orbital perturbations; but it is judged that only the effects of air resistance and of oblateness can surely be separated from the others.

J. A. O'Keefe (Chevy Chase, Md.)

751:

Abalakin, V. K. On the stability of libration points near the gravitating ellipsoid in rotation. Byull. Inst. Teoret. Astr. **6** (1957), 543–549. (Russian. English summary)

In the present paper the stability of the libration points in the case of a particle moving under the gravitational attraction of a rotating ellipsoid has been discussed. The libration points which lie on the extensions of the minor axis of the equatorial section of the ellipsoid have been found to be stable by the first approximation, while these disposing in direction of the major axis are unstable. Three families of periodic orbits are obtained, the sizes of which depend on one parameter. In addition, the one-parametric family of asymptotic solutions has been also derived. *Author's summary*

752:
Batrakov, Yu. V. The periodical motions of a particle under gravitation of a rotating three-axial ellipsoid. Byull. Inst. Teoret. Astr. **6** (1957), 524–542. (Russian. English summary)

The article discusses the motion of a particle of zero mass under the action of gravity of a homogeneous three-axial ellipsoid, the latter differing but little from a sphere of the same volume and rotating uniformly about one of the axes of symmetry. The possibility of periodical solutions of the third, second and first sort in this problem has been established. *Author's summary*

753:
Žongolovič, I. D. Potentiel de l'attraction terrestre. Byull. Inst. Teoret. Astr. **6** (1957), 505–523. (Russian. French summary)

On donne le développement du potentiel de l'attraction de la Terre aux termes du deuxième ordre de l'aplatissement près. On cherche a déterminer les valeurs numeriques des coefficients de ce développement en utilisant les résultats obtenus auparavant par l'auteur pour le champ extérieur de la gravitation terrestre. Le développement obtenu peut être utilisé dans l'étude du mouvement des corps dans le voisinage de la Terre. *Résumé de l'auteur*

754:
García, Godofredo. Motion of the spiral nebulae. Actas Acad. Ci. Lima **20** (1957), no. 4, 8–34. (Spanish)

755:
***Хильми, Г. Ф. [Hil'mi, G. F.] Качественные методы в проблеме n тел. [Qualitative methods for the n-body problem].** Izdat. Akad. Nauk SSSR, Moscow, 1958. 123 pp. (1 insert). 295 rubles.

An elegant development in terms presupposing little, if anything, beyond ordinary calculus. The first chapter develops the general integrals of the system of equations; the second applies dimensional analysis to obtain the form of an inequality expressing a condition for dispersal. The third and fourth chapters contain the main results, applying the "method of continuous induction" and the "method of invariant measure" (the theorem of Liouville). A concluding chapter compares parameters of a system before and after a change in the number of bodies in the system. *A. S. Householder* (Oak Ridge, Tenn.)

756:
Scorgie, G. C. On free motion in the gravitational field of the earth. Quart. J. Mech. Appl. Math. **10** (1957), 494–499.

As stated by the author, the present article is concerned with the solution of the basic problem of the free motion of a particle in the earth's gravitational field when account is taken of the ellipticity of the figure of the earth, that is, of the departure of the external field of gravity from a force varying inversely as the square of the distance from a fixed center. An approximate solution is obtained by use of special polar coordinates q^n where $n=1, 2, 3$. The methods used are those of the tensor calculus. By means of the Hamilton-Jacobi equation and the theory of contact transformations, the motion is expressed in terms of six constants, namely, E, the total energy of unit mass; m, the angular momentum of unit mass about the earth's axis; t_a, the time of passage through apogee; Q^2, the apogee distance; Q_0^2, the value of q^2 for which q^3 vanishes; and a constant n. By means of the Serret-Frenet equations, the torsion σ of such a trajectory is evaluated. In the final part of the paper, numerical estimates are given. *J. De Cicco* (Chicago, Ill.)

757:
McCrea, W. H. The formation of population I stars. I. Gravitational contraction. Monthly Not. Roy. Astr. Soc. **117** (1957), 562–578.

The initiation of gravitational collapse of a spherical mass of gas is considered starting from a general form of the virial theorem. The gas is supposed to have a boundary on which the pressure is given. Isothermal masses of gas are dealt with and the conditions for gravitational collapse are determined; for example, it is shown that for a given total mass and boundary pressure there is a minimum temperature below which no equilibrium state is possible. The theory is compared with the exact theory of Ebert in which a precise internal density distribution is established. This exact theory, though an idealized one, is shown to give good approximations to the results expected in more realistic conditions. Finally, distributions of matter with plane and with cylindrical symmetry are considered for elucidatory purposes.
G. C. McVittie (Urbana, Ill.)

758:
Lindquist, Richard W.; and Wheeler, John A. Dynamics of a lattice universe by the Schwarzschild-cell method. Rev. Mod. Phys. **29** (1957), 432–443.

The idea is developed of a lattice universe consisting of mass concentrations, represented by Schwarzschild's metric, with equal masses arranged in a regular lattice in a closed space. Special interest is attached to the case of 600 such masses, the largest number that can be arranged in a regular lattice. The equation of motion of the mass at the center of a cell appears as a dynamic condition on the boundary of the cell, and is a consequence of the field equations. In the case when closed space is divided up into infinitely many, infinitely small cells, one may pass from the lattice universe to the dust-filled universe of Friedmann. Indeed, the calculated radius of maximum expansion for the case of 600 particles agrees to 1.2% with the result of Friedmann.
L. Infeld (Warsaw)

GEOPHYSICS

See also 517, 538, 540.

759:
***Scheidegger, Adrian E. Principles of geodynamics.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xi +280 pp. DM 49.60.

This book is an assemblage of a vast number of geophysical facts and theories relating to the sub-atmospheric Earth, and an attempt to evaluate them. There are sections, averaging some six pages or so, on such widely different topics as geological evolution, meteor craters, volcanoes, gravity anomalies, seismic data, thermal data, geochemical data, magnetisation of rocks, polar wandering, the figure of the Earth, finite strain, infinitesimal elasticity theory, finite strain theory, imperfect elasticity, continental drift, the thermal contraction theory, convection current theories — to name but some of the topics.

The author claims that all other geophysical books have been written to advance one or other of many competing hypotheses as the "true" one, and describes his approach as "agnostic". He refers to the book as presenting a "synoptic view of the subject, much in the same way as one might present a synoptic view of the world's philosophies".

The result is a book which contains a quantity of useful and up-to-date material on matters of current geophysical discussion and controversy, as well as discussions of older theories. Its chief value lies in the bird's eye view it gives of a sizable part of geophysics; it may help to widen the horizons of some students of geophysics. Several of its evaluations will interest the research geophysicist, as will also the clarity and freshness of approach in some of the sections dealing with the more recent work. The reader, however, would be advised to check the detail in some of discussions against deeper accounts in standard papers and treatises.

The book is, by its nature, of interest to geophysicists rather than to mathematicians in general, since its wide coverage forbids the presence of many sustained formal arguments exceeding a page or so.

K. E. Bullen (Sydney)

760:

Voĭt, S. S. Propagation of confluent waves from a strait into an open basin. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958,** 486–496. (Russian)

761:

Thompson, Philip Duncan. Uncertainty of initial state as a factor in the predictability of large scale atmospheric flow patterns. Tellus **9** (1957), 275–295.

This paper deals with the gradual growth of errors of prediction, due to errors in an initial state which is reconstructed from measurements at a finite number of points in the atmosphere.

Assuming that, given an initial state, the solution of the Navier-Stokes equations can be computed by approximate numerical methods with a numerical error less than any preassigned value, the author considers two different initial states; one the "true-state" and the other the "reconstructed state", consisting of the true state plus a field of analysis error that is random with respect to the true state. The difference between the two solutions, at a time t, is a measure of the essential unpredictability of the atmosphere. In particular, if the average magnitude of such differences approaches the average error of "guessing", the atmosphere has become essentially unpredictable beyond that time.

The method developed here does not solve, in fact, this general problem but deals only with a simplified model of the atmosphere, the "quasi nondivergent" model proposed by H. L. Kuo.

To ψ_0 and ψ_0+R_0, stream functions of the initial state and the reconstructed initial state (R_0 is a random function), correspond at a time t the stream function ψ and $\psi+R$; the author defines the statistical measure of error E as the mean square value of the gradient of R taken over the whole area under consideration; his main result is an explicit expression of the time-derivatives of E in terms of ψ and R at the current time, for a nondivergent barotropic and a two-level baroclinic flow. From these examples of the rate at which analysis error contaminates a forecast, the author makes a crude estimate of the maximum time-range of predictability; using hypotheses (which certainly need further discussion) he gives $\tau=7.7$ days.

J. Kampé de Fériet (Lille)

762:

Yanovskaya, T. B. On the determination of the dynamic parameters of the epicenter of an earthquake from observations of surface waves. I. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958,** 289–301. (Russian)

763:

Evernden, J. F. Finite strain theory and the earth's interior. Geophys. J. **1** (1958), 1–8.

Let ρ_0 and K_0 denote density and incompressibility at zero pressure. Using a pressure-strain equation of Brillouin and certain geophysical assumptions, the author derives values of ρ_0 and K_0 for the Earth's lower mantle and outer core which are compatible with the density and pressure distributions of Bullen's Earth model A. (K. E. Bullen, Introduction to the theory of seismology, University Press, Cambridge, 1953; MR **15,** 373; pp. 212–222). Geophysical implications are briefly discussed.

An appendix shows the equivalence of certain equations in the Poincaré-Brillouin and the Murnaghan-Birch approaches to finite-strain theory.

K. E. Bullen (Sydney)

764:

Bulah, E. G. On some criteria for verification of the interpretation of gravitational and magnetic anomalies. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957,** 1173–1176. (Russian)

Considering the two-dimensional case, when the excess-masses, the cause of the anomaly, fill up an infinite horizontal cylinder (parallel to $\overline{oy}$), the author deduces the well-known classical expressions for the moments of orders zero and one of g, $\partial q/\partial x$ as well as of Z and H of the magnetic anomaly.

E. Kogbetliantz (New York, N.Y.)

765:

Silva, Giovanni. Sulla determinazione degli errori di lettura di una livella dovuti a imperfezioni della costruzione Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 257–268.

A method of determining and taking account of deviations from circularity in the longitudinal section of a high precision level is discussed.

A. Marussi (Trieste)

766:

***Sakatow, P. S. Lehrbuch der höheren Geodäsie. Aus dem Russischen übersetzt von H. A. Corazza.** VEB Verlag Technik, Berlin, 1957. 467 pp. DM 45.00.

The book is made up of lectures on geodesy given in Advanced Technical Courses held in the Universities of the U.S.S.R., and more recently in those of East Germany. The subject is dealt with principally from the point of view of applied science and includes the geometry of the ellipsoid, Gauss's conformal representation, the gravimetric procedures for the determination of the geoid and the deflections of the vertical, the techniques of geodetic astronomy and the adjustment of great triangulation networks, account being taken of the discrepancy between geoid and ellipsoid.

The book is clearly written and the treatment exhaustive, though not entirely free from inexactitudes, for example on page 156, where Gauss's representation is identified with a central cylindrical projection.

The most interesting chapter, which represents in part an original contribution in the field, deals with astro-gravimetric levelling according to Molodienski, which is based on the Stokes-Vening Meinesz formula. Free air anomalies are used; the formula for normal gravity is that

given by Helmert, which, strictly speaking, is incompatible with the use of the ellipsoid as surface of reference.

Throughout the whole work, emphasis is placed on the practical and theoretical contributions made by Russian geodesists. In addition, there is ample historical material on the work in the field of geodesy carried out within the vast territories of the Soviet Union.

A. Marussi (Trieste)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 352, 418.

767:

***Arrow, Kenneth J.; Karlin Samuel; and Scarf, Herbert. Studies in the mathematical theory of inventory and production. With contributions by M. J. Beckmann, J. Gessford and R. F. Muth.** Stanford Mathematical Studies in the Social Sciences, I. Stanford University Press, Stanford, Calif., 1958. xi+340 pp. $8.75.

Two of the most interesting classes of dynamic programming processes, viewed from the vantage points of both analysis and application, are those of inventory control and production smoothing.

The typical inventory control process may be described in the following general terms. We possess various quantities of different items for which there is a demand of stochastic nature from time to time. Since there is a penalty of some type attached to not being able to satisfy this demand, at various stages additional quantities of these items are ordered, at costs dependent upon types and quantities ordered, the times of ordering, the rate of delivery desired, and other factors as well. The problem is to determine the ordering policies which are optimal with respect to preassigned measures of efficiency.

Production smoothing processes are of quite similar structure, with the additional feature that there may or may not be costs attached to varying rates of production and rates of ordering.

Both are representative of the kinds of multi-stage decision processes that arise in economic and industrial activities.

In order to assess the role of the present volume in the continuing research in these fields, let us briefly review the very recent history of the optimal inventory problem. In the first place, there is the recognition of this decision process as a problem worthy of attention for its own sake, apart from the larger activity within which it is always imbedded. For a history of this phase, see the book by T. Whitin [The theory of inventory management, Princeton Univ. Press, 1953], which also contains a discussion of various preliminary mathematical attacks upon particular aspects of the general problem.

The next phase involves the analytical formulation using the techniques of the modern approach to the analysis of stochastic processes. The classic paper of Arrow, Harris and Marschak, [Econometrica **19** (1951), 250–272; MR **13**, 368], inaugurates the use of functional equations to describe the process.

From this point the road forks. To begin with, there are essential questions pertaining to the existence and uniqueness of solution of the nonlinear functional equations obtained in this fashion. These were first treated in the paper by Dvoretzky, Kiefer and Wolfowitz [Econometrica **20** (1952), 187–222, 450–466; MR **13**, 856; **14**, 301]. Only after these points are settled can one be sure of a one-to-one correspondence between the optimal policies of the original process and the solutions of the defining equations.

Secondly, there is the basic problem of studying not only optimal policies, but also sub-optimal policies. Within a class of simple feasible policies, we wish to determine the most efficient. This road is followed by Arrow, Harris and Marschak in the paper referred to above, and the inverse problem was studied by Dvoretzky, Kiefer and Wolfowitz.

Thirdly, the functional equation can be utilized to determine the structure of optimal policies as a function of the structure of the cost and penalty functions, and also of the probability distribution occurring. This study was initiated by Bellman, Glicksberg and Gross [Management Sci. **2** (1955), 83–104; MR **17**, 641; see also R. Bellman, Dynamic programming, Princeton Univ. Press, 1957; MR **19**, 820; Ch. 5.]

In the book under review we find a continuation of this effort, carried out quite elegantly and in great detail, and, in addition, a continuation of the description of the outcomes of various approximate policies. This last requires the use of the tools of recent renewal and queuing theory.

One of the most interesting chapters is that devoted to timelag processes, which is to say those in which there is an appreciable delay between ordering and delivery. Under the assumption that backlogs occur, it is shown that the number of state variables required to describe the process may be drastically reduced. Consequently, the results obtained by Bellman, Glicksberg and Gross in the paper cited above may be extended to this case. This furnishes a ready approximation to the solution of more complicated processes.

Turning to the study of production processes over time, a pioneering work is that of Pierre Massé [Les reserves et la régulation de l'avenir dans la vie économique, Hermann, Paris, 1946; MR **10**, 52, 53]. A number of interesting processes give rise via simplified mathematical models to variational problems of the following type: "Maximize a functional of the form $J(u)=\int_0^T g(u, u', t)dt$, subject to combinations of constraints of the type $a_1 \leq u(t) \leq a_2$, $a_3 \leq u'(t) \leq a_4$, $\int_0^T h(u, u')dt \leq a_5$." The presence of the constraints renders some of these problems of far greater difficulty than might be supposed upon first observation. Let us note in passing that questions of this nature arise with great frequency in the study of current engineering control processes.

The present series of studies contains an analytic treatment of some production processes and of some related problems arising in the consideration of hydroelectric systems. References are given to other work by Hohn, Modigliani and Simon, by Koopmans, by Karush and Vaszonyi, and by Bellman, Glicksberg and Gross. A good deal of unpublished work on these problems was carried out at RAND from 1948 on by Paxson, Danskin, and Fleming, stimulated by the book of Massé referred to above. Further study of variational problems with constraints may be found in Bellman, Glicksberg and Gross [Rend. Circ. Mat. Palermo (2) **3** (1954), 363–397; MR **16**, 1127] and in Bellman, Fleming and Widder [Ann. Mat. Pura Appl. (4) **41** (1956), 301–323; MR **18**, 51].

The principal authors, as well as Beckmann, Gessford and Muth, have maintained a high mathematical level with no sacrifice of clarity. All of the studies are worth-

while starting points for further research in this fundamental domain, and the introductory expository section is particularly valuable in enabling the reader to understand the origin of the following analysis.

There are, however, some ways in which the level of the book could be raised even higher. In the first place, it would have been worthwhile to have included some computational results indicating the dependence of optimal policies, minimum costs and maximum profits upon the different parameters entering into the mathematical models. Along these lines, it might also have been helpful to point out that the functional equation technique of dynamic programming furnishes computational solutions to the variational problems described in the preceding sections in a quite simple fashion without the aid of any assumptions concerning convexity, concavity and so forth; see Chapter 9 of the book referred to previously.

Secondly, it should be emphasized that one of the most important aspects of economic control processes of this type is that of predicting the future demand on the basis of the observed past demand. We thus encounter "learning" or "correlative" processes of the type that have come into prominence with the development of sequential analysis by Wald, and by Wolfowitz, Blackwell, Girshick, Karlin, and others. This fundamental problem was attacked in relation to inventory processes in the second of the papers by Dvoretzky, Kiefer and Wolfowitz referred to above.

Considering the size of the book and the amount of material it contains, it is understandable, however, that the authors felt that a line must be drawn somewhere. Perhaps, we can look forward to a second set of studies devoted exclusively to these deeper problems. In any case, the book is certainly to be recommended to anyone working in the fields of mathematical economics and operations research. *R. Bellman* (Santa Monica, Calif.)

768:

*Goldberg, Samuel. **Introduction to difference equations, with illustrative examples from economics, psychology, and sociology.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xii+260 pp. $6.75.

This book gives an elementary but careful treatment of finite differences and difference equations, with emphasis on their applications to the social sciences. The book was written primarily for social scientists and therefore includes brief developments of relevant mathematical topics such as functions, limits and matrices.

Chapter 0 provides an informal description of two examples: a learning model and a dynamic model from economics. Chapter 1 deals with finite differences, Chapter 2 provides an introduction to difference equations, and Chapter 3 develops the general linear difference equation with constant coefficients. Particular emphasis is placed on the limiting behavior of the solutions. Chapter 4 deals with selected topics relevant to difference equations. These include equilibrium and stability, cobweb cycles, characteristic values, the use of generating functions including a brief introduction to their use in probability theory, and the use of matrix methods in solving simultaneous difference equations. Analogies with calculus and differential equations are brought out in starred sections of the appropriate chapter.

Of interest to the mathematician is the fact that the book is everywhere dense with examples taken from actual research papers in the social sciences. The book provides a very efficient way to survey the uses of finite calculus in the social sciences.

An excellent set of exercises is provided. These include further developments of the models introduced in the text. In addition to a large number of references in the text, a selected list of references is given at the end of the book. *J. L. Snell* (Hanover, N.H.)

769:

Kantorovič, L. V. Methods of analyzing some extremum problems concerned with industrial programs. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 441–444. (Russian)

Each production method π_s ($s=1, \cdots, N$) is associated with three vectors $X^{(s)}=(x_1^{(s)}, \cdots, x_l^{(s)})$, $Y^s=(y_1^{(s)}, \cdots, y_m^{(s)})$, $Z^s=(z_1^{(s)}, \cdots, z_n^{(s)})$ denoting the quantities produced (or, if negative, used) of final goods, intermediate goods and production factors, respectively. An admissible plan is a vector $P=(p_1, \cdots, p_n)$, $p_s \geqq 0$; more specifically, one assumes linearity, i.e., associates P with the triple of vectors $X=\sum_1^N p_s X^{(s)}$, $Y=\sum_1^N p_s Y^{(s)}$, $Z=\sum_1^N p_s Z^{(s)}$. $P=\bar{P}$ is called optimal if $X \geqq kX_0$, $Y \geqq 0$, $Z \geqq -Z_0$, where X_0 (the desired bill of final goods) and Z_0 (limit on primary resources) are given positive numbers, and k has its maximum value. The following theorem is proved:

Suppose that $X \geqq 0$, $Y \geqq 0$, $Z \geqq 0$ only if all $p_s=0$. Then an optimal plan $P=\bar{P}$ exists and is associated with a system of multipliers (called valuations)

$$\Xi=(\xi_1, \cdots, \xi_l) \geqq 0, \quad H=(\eta_1, \cdots, \eta_m) \geqq 0,$$

$$Z=(\zeta_1, \cdots, \zeta_n) \geqq 0,$$

such that, denoting by $(\Xi, X^{(s)})$ the sum $\sum_{i=1}^{l} \xi_i x_i$, etc.,

(1) $(\Xi, X^s)+(H, Y^{(s)})+(Z, Z^{(s)}) \leqq 0$, $s=1, \cdots, N$;

(2) $(\Xi, X^s)+(H, Y^{(s)})+(Z, Z^{(s)})=0$ if $\bar{p}_s>0$.

Conversely: if a plan $\bar{P}$ is associated with the triple

(3) $$\bar{X}=kX_0, \quad \bar{Y}=0, \quad \bar{Z}=-Z_0,$$

and there exist multipliers satisfying (1) and (2), then $\bar{P}$ is optimal; this remains true if the equality signs in (3) are replaced by "$\geqq$", provided the multipliers corresponding to "$>$" vanish.

Three methods of computing optimal plans by successive approximation using valuation vectors are mentioned.

The problem is also extended to the case when production methods change over time t, by re-defining $X^{(s)}$, $Y^{(s)}$, $Z^{(s)}$ as matrices $[x_{i,t}^{(s)}]$, $[y_{j,t}^{(s)}]$, $[z_{k,t}^{(s)}]$, with corresponding multiplier matrices $[\xi_{i,t}]$, $[\eta_{i,t}]$, $[\zeta_{i,t}]$, $t=1, \cdots, T$. In particular, if each production method yields only one final good, and the multipliers are normalized by requiring $\sum_t \xi_{t,t'}=1$, $\xi_{t,t}=\lambda_t \xi_{t,t'}$, etc., $t=1, \cdots, T$, condition (1) becomes

(1') $$\sum_t \lambda_t \Big(\sum_i \xi_{i,t'} x_{i,t}^{(s)}+\sum_j \eta_{j,t'} y_{j,t}^{(s)}+\sum_k \zeta_{k,t'} z_{k,t}^{(s)}\Big) \leqq 0,$$

and λ_t^{-1} can be interpreted as a general rate of growth.

The paper summarizes previous work of the author (dating from as early as 1939) and his collaborators, M. K. Gavurin, V. A. Zalgaller, G. Sh. Rubinstein, and refers also to some non-Russian authors: D. Gale, G. Dantzig, T. C. Koopmans, W. Leontief.

J. Marschak (Pittsburgh, Pa.)

770:

Chandler, Robert E.; Herman, Robert; and Montroll, Elliott W. Traffic dynamics: studies in car following. Operations Res. 6 (1958), 165–184.

The authors consider various dynamic models for the motion of a sequence of cars. The trajectory of each car is assumed to depend upon the motion of the car ahead at some earlier time and to satisfy a linear differential difference equation. They investigate the propagation of a disturbance from one car to the next and demonstrate that under certain conditions one may obtain an instability in the sense that the amplitude of the disturbance increases as it propagates down the line of cars. Some experiments were performed to show that the acceleration of a car is nearly proportional to the difference between its velocity and that of the car ahead, measured at a time approximately 1.5 seconds earlier. These data indicate that the drivers' pattern of behaviour is very close to the limit of stability. *G. Newell* (Stockholm)

771:

Bílý, Josef. Measurement of the lifetime of equipment under operational conditions. Pokroky Mat. Fys. Astr. 2 (1957), 14–18. (Czech)

The author describes a method of comparing the durations of life of different types of equipment serving the same purpose without keeping records of individual life lengths; these methods have been considered by Goodman [J. Amer. Statist. Assoc. 48 (1953), 503–530]. The author shows, for the case of three types, several replacement policies which are generalizations of policies consisting of the replacement of failed items by items of another type (so called "switch" policies) and of the policy involving a random choice of the type which is to replace the failed item. Various procedures are compared and analogous procedures for more than three types are indicated. A constant intensity of removal and a continuous replacement is assumed throughout (an item is replaced immediately after it has failed). *J. Janko* (Prague)

772:

Zwinggi, Ernst. Due procedimenti per determinare i premi addizionali per rischi aggravati. Giorn. Ist. Ital. Attuari 19 (1956), 16–21.

Approximate determination of extra premiums to the endowment assurance in the case of under-average lives is given. The case when the increased probability of death is expressed in a multiplicative form $q_x^{(\gamma)}=(1-\gamma)q_x$ is considered first. In a sufficiently broad interval the extra premium varies linearly with $\gamma\leq 1$. Secondly, the case where the increased probability of death is expressed in an additive form $q_x=q_x+\gamma$ is considered. The extra premium is determined in an especially simple way and also depends linearly on γ. The degree of approximation is numerically illustrated in both cases by the Swiss mortality table SM 1939/44, $2\frac{1}{2}\%$. *J. Janko* (Prague)

773:

Prager, William. On warehousing problems. Operations Res. 5 (1957), 504–512.

A linear programming problem originally formulated by Cahn and subsequently treated by Charnes and Cooper is shown to be a generalized Hitchcock-Koopmans transportation problem. This identification permits the use of a simple algorithm for solution, and also the inclusion of storage costs. As the author points out, Dantzig has also noted the connection between the warehousing and transportation problem. A numerical example is given. *R. Bellman* (Santa Monica, Calif.)

774:

Dreyfus, Stuart E. An analytic solution of the warehouse problem. Management Sci. 4 (1957), 99–104.

The warehousing problem discussed in the preceding review has also been treated by means of the theory of dynamic programming [Bellman, Management Sci. 2 (1956), 272–275; MR 19, 931]. In this paper the author shows that the functional equation derived in the cited reference for the purposes of computational solution can actually be resolved analytically. A numerical example is given to indicate how this solution can be used.
R. Bellman (Santa Monica, Calif.)

775:

Barankin, E. W.; and Dorfman, R. On quadratic programming. Univ. California Publ. Statist. 2 (1958), 285–318.

This paper reports the authors' work on the "Quadratic programming" problem, which, initiated several years ago, has stimulated considerable research into the field. They pose the problem, called $P(A, \beta, a, b)$, as: Given the n-by-n positive semidefinite matrix A, the n-vector β, the m-by-n matrix a, and the m-vector b, determine an n-vector x which shall maximize the function $g(x)=\beta x-(Ax, x)$ under the linear inequality constraints $x\geq 0$, $ax=b$.

The results of the first section of the paper are summarized in Theorem 1.8: In the problem $P(A, \beta, a, b)$, the function g is concave; if the problem is solvable, then the set of all solutions x is closed and is the intersection of the set determined by the constraints with any set of the form $\xi+z$, where ξ is one solution of the problem and z ranges over the intersection of the null-space of A with the manifold perpendicular to β.

The following section establishes the insensitivity of the solution of $P(A, \beta, a, b)$ to certain small changes of its data: Assume $P(A, \beta, a, b)$ solvable; let c^0 be its supremum and x^0 its solution of smallest norm; let c', x' denote the corresponding quantities for the problem $P(A+\delta I, \beta, a, b)$ and c^*, x^* those for $P(A, \beta, a, b^*)$. Then (Theorem 2.1) $\lim_{\delta\to 0+} c'=c^0$, $\lim_{\delta\to 0+} x'=x^0$, and (Theorem 2.2) $\lim_{\beta^*\to\beta} c^*=c^0$, $\lim_{\beta^*\to\beta} x^*=x^0$.

The problem P is next transformed, by use of the generalized Lagrange multipliers of Kuhn and Tucker [Nonlinear programming, Proceedings of the second Berkeley symposium on math. statistics and probability, Univ. Calif. Press, Berkeley, 1951, 481–492; MR 13, 855], into the problem $Q(A, \beta, a, b)$: To find vectors x, u, v, s (of appropriate dimension) satisfying the linear relations $2Ax+ua-v=\beta$, $ax+s=b$, $x, u, v, s\geq 0$, and which achieve the minimum, zero, of $T(x, u)=vx+us$. Using the results of the previous section, if the original problem P was solvable, a new problem P' whose solution is prescribedly close to the solution of P can be constructed for which A' is nonsingular and such that any solution of the linear relations of Q' has exactly $m+n$ nonvanishing quantities from the $2(m+n)$ components of x, u, v, s.

In the final section, this remarkable property of the reformulated perturbed problem Q' is used to show that a solution x of P is obtained as the x-part of a basic solution (x, u, v, s), in the sense of the simplex method, of the relations of the problem Q. The question of whether $T(x, u)$ can be reduced to zero by means of simplex method changes-of-basis is studied; examples show that neither T itself, nor the count of the number of non-zero terms in

the inner products vx, us, can always be reduced by a single change-of-basis. The paper concludes by posing the question whether either of these quantities or some combination of them might be reduced to zero by an iterative process, say, involving two changes-of-basis.

P. Wolfe (Santa Monica, Calif.)

776:

Taylor, R. J.; and Thompson, S. P. On a certain problem in linear programming. Naval Res. Logist. Quart. **5** (1958), 171–187.

If $G=\sum_{j=1}^{N}\sum_{i=1}^{n_j}x_j(i)a_j(i)$ and $w=\sum_{j=1}^{N}\sum_{i=1}^{n_j}w_ja_j(i)$ for $x_j(i)>0$, $w_j>0$, and $0\leqq a_j(n_j)\leqq a_j(n_j-1)\leqq\cdots\leqq a_j(1)\leqq 1$, the authors prove a theorem to give an explicit algebraic statement of the values of a_j which maximise G subject to the constraint

$$0\leqq w\leqq\sum_{j=1}^{N}n_jw_j.$$

They define for each j and $n_j\geqq\alpha>\beta\geqq 0$, $X_j(\alpha,\beta)=\sum_{i=\beta+1}^{\alpha}x_j(i)$ and $C_j(\alpha,\beta)=X_j(\alpha,\beta)/w_j(\alpha-\beta)$, and they associate with each j integers $s_j=1, 2, \cdots$, and $k_j(s_j)>k_j(0)=0$, such that for an arbitrary i in $k_j(s_j-1)<i\leqq n_j$, $C_j[k_j(s_j), k_j(s_j-1)]\geqq C_j[i, k_j(s_j-1)]$. Also $k_j(S_j)=n_j$.

The theorem states that for $W(q_0)<w\leqq w(q_0+1)$, where q_0 is selected arbitrarily from the range $1\leqq q\leqq\sum_{j=1}^{N}S_j$:

$$\max_{a_j(i)}G[a_j(i)]=\frac{W-\sum_j w_jk(p_j)}{w_r[k(p_r+1)-k(p_r)]}X_r[k(p_r+1), k(p_r)]+\sum_{j=1}^{N}X_j[k(p_j), 0],$$

and

$$a_r(i)=\frac{W-\sum_j w_jk(p_j)}{w_r[k(p_r+1)-k(p_r)]}\quad\text{for } k(p_r)<i\leqq k(p_r+1),$$

$$a_j(i)=\begin{cases}0 & \text{for } k(p_r+1)<i\leqq n_r,\\ 1 & \text{for } 1\leqq i\leqq k(p_j) \text{ and all } j,\\ 0 & \text{for } k(p_j)<i\leqq n_j \text{ and all } j\neq r,\end{cases}$$

where $r=j(q_0+1)$.

The process is a disguised form of the standard contracted process, and the proof, though complicated, is at an elementary level. No mention is made of any practical experiences in the use of the theorem when applied to calculations on a high-speed computor and no discussion is given of the difficulties to be encountered or of the likely size of accumulated errors in the results.

L. J. Slater (Cambridge, England)

777:

Manne, Alan S. A target-assignment problem. Operations Res. **6** (1958), 346–351.

The problem considered is that of finding an effective computational procedure for determining values of x_{ij} that satisfy the conditions (a) $\sum_{i=1}^{n_j}x_{ij}=1$, (b) $x_{ij}\geqq 0$, (c) $y_j=\sum_{i=1}^{n_j}x_{ij}$, and that minimize the quantity $M=\sum_{j=1}^{n_j}a_j(1-p_j)^{y_j}$, where $0<p_j<1$ and $a_j>0$ are given constants. This problem is shown to be a special case of the transportation problem of linear programming, and an interesting numerical example is calculated to show how the main problem is solved easily by the simplex method. The problem is offered as an approximation to the target-assignment problem proposed by M. M. Flood, consisting of conditions (a) and (b) and minimization of the quantity $F=\sum_{j=1}^{n_j}a_j\prod_{i=1}^{n_j}(1-p_{ij}x_{ij})$, where $0<p_{ij}<1$, a problem as yet unsolved. The author notes that his methods apply equally well if the quantity to be minimized is $D=\sum_{j=1}^{n_j}a_j\prod_{i=1}^{n_j}(1-p_{ij})^{x_{ij}}$ and, as George Dantzig suggested, the quantity D is the same as F for x_{ij} restricted to the values 0 or 1. The author suggests that minimization of D may yield good approximations for F, and notes that the minimum value of D is a lower bound for F. It is easily shown that integral values of x_{ij} will minimize F, as the author notes is the cases for M but not for D.

M. Flood (Ann Arbor, Mich.)

778:

***Vajda, S. Readings in linear programming.** John Wiley & Sons, Inc., New York, 1958. vii+99 pp. $3.00.

A collection of numerical examples illustrating 16 of the best known linear programming models and serving as a general introduction.

M. J. Beckmann (New Haven, Conn.)

779:

***Scarf, H. E.; and Shapley, L. S. Games with partial information.** Contributions to the theory of games, vol. 3, pp. 213–229. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

This paper discusses games in which each player is informed of his opponent's moves a fixed amount of time after they are made. The authors introduce certain "generalized subgames" which begin with a chance move, the results being partially withheld. The authors show that if the value of the generalized subgame exists, then the solution of a certain recursive relation yields the value and optimal strategies of the original game. They also show that a sufficient condition for the existence of the game value is that there be a finite number of choices at each move.

M. Dresher (Pacific Palisades, Calif.)

780:

Stone, Jeremy J. An experiment in bargaining games. Econometrica **26** (1958), 286–296.

BIOLOGY AND SOCIOLOGY

See also 768.

781:

Schneider, Berthold. Stochastische Grundlagen für eine statistische Auswertung der EEG-Kurven. Mitt. Math. Sem. Giessen no. **56** (1958), 49 pp.

The author critically discusses previous mathematical models for nerve networks, in particular, the suggestion of H. v. Schelling that simple branching processes be used [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1944, no. 6; MR **8**, 214] and the theory of A. Shimble and A. Rapoport [Bull. Math. Biophys. **10** (1948), 41–55; MR **9**, 521]. He advocates as a model a vector-valued random set function $P(B)$. Each small element of volume in a three-dimensional organ such as the brain produces a small electric field vector, and $P(B)$ is the sum of the vectors for the small pieces constituting B. The basis for this assumption is work of F. Stallmann on the electrocardiogram [Archiv für Kreislaufforschung **25** (1957), 291–323.] There is some discussion of random set functions and of the problem of estimating the spectrum of the stationary process represented by the electroencephalogram.

T. E. Harris (Santa Monica, Calif.)

782:

***Lotka, Alfred J. Elements of mathematical biology. (formerly published under the title Elements of Physical Biology).** Dover Publications, Inc., New York, N. Y., 1958. xxx+465 pp. $2.45.

This book was originally published in 1925. The new

edition is unchanged apart from minor corrections and the addition of a list of the author's papers. It is really a book on the theory of biological systems, written from a very original viewpoint, at times philosophical in character, and frequently illuminated by mathematical argument. The mathematical approach is almost entirely by way of systems of differential equations, usually treated rather generally. After an introduction on the nature of evolution, there follow three sections. 1. Kinetics, which embraces models for population growth and interaction (always from a deterministic point of view). 2. Statics, concerned with equilibria, slowly evolving systems and stable cyclic movements. 3. Dynamics, the study of energy transformations. Throughout the book the author brings to bear a formidable battery of relevant facts and theories from almost every branch of science and human behaviour. *P. Armitage* (London)

783:

Gulliksen, Harold; and Tukey, John W. Reliability for the law of comparative judgment. Psychometrika **23** (1958), 95–110.

From authors' summary:" A variance-components analysis is presented for paired comparisons in terms of three components: s, the scale value of the stimuli; d, a deviation from the linear model specified by the law of comparative judgement; and b, a binomial error component. Estimates are given for each of the three variances σ_s^2, σ_d^2, and σ_b^2. Several coefficients, analogous to reliability coefficients, based on these three variances are indicated. The techniques are illustrated in a replicated comparison of handwriting specimens."

F. C. Andrews (Eugene, Ore.)

784:

Haimovici, A. Introduction à l'étude mathématique des associations biologiques. Gaz. Mat. Fiz. Ser. A. **4** (1958), 193–204. (Romanian. French and Russian summaries)

The author gives a good expository account of results obtained by V. Volterra [Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier-Villars, Paris, 1931; Variazioni e fluttuazioni del numero d'individui in specie animali conviventi, Ferrari, Venice, 1927; Les associations biologiques au point de vue mathématique, Hermann, Paris, 1935]. He uses Volterra's results to discuss the case of two species, one herbivorous, and the other carnivorous, which have the same habitat.

R. G. Stanton (Waterloo, Ont.)

785:

Bartholomay, Anthony F. On the linear birth and death processes of biology as Markoff chains. Bull. Math. Biophys. **20** (1958), 97–118.

Expository paper.

T. E. Harris (Santa Monica, Calif.)

786:

Kimura, Motoo. Some problems of stochastic processes in genetics. Ann. Math. Statist. **28** (1957), 882–901.

A review paper containing several new results. The author discusses: (1) The probability of the appearance of a mutant chromosome, when a chromosome is composed of sub-units, some of which may be mutant; (2) the senescence of Paramecium due to accumulation of chromosome aberrations; (3) natural selection of genes in a finite population; results are obtained for the cases when one gene has a selective advantage, and dominance is either absent or complete; (4) the probability of fixation of a mutant gene; (5) the spread of a mutant gene in a population distributed uniformly over the plane.

P. Whittle (Wellington)

787:

***Stephan, Frederick F.; and McCarthy, Philip J. Sampling opinions: an analysis of survey procedure.** Wiley Publications in Statistics. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. xxi+451 pp. $12.00.

This book is a thorough and competent discussion of extra-mathematical aspects of the surveying of human populations. Theoretical insights, and extensive empirical information about accuracy, cost, and practicality of survey techniques make the book useful, not only to those to whom it is mainly directed, but also to some mathematical statisticians. *L. J. Savage* (Rome)

INFORMATION AND COMMUNICATION THEORY

See also 350, 351.

788:

Rice, S. O. Distribution of the duration of fades in radio transmission: Gaussian noise model. Bell System Tech. J. **37** (1958), 581–635.

Mathematically, this paper is devoted to the problem of obtaining approximations to the distribution of the periods τ between crossings of fixed levels of certain stationary stochastic processes. The processes considered are of type (a) $I(t)$, a Gaussian process of mean 0, and (b) $R(t)$, the envelope of a narrow-band $I(t)$-process with symmetric spectral density.

The expected values of τ are known for (a) and (b). The author considers approximations for $F(u, I)=\Pr\{$a period during which $I(t)>I$ exceeds $\bar{\tau}u\}$, and the corresponding distribution for $R(t)$. Many of the expressions are too complicated to be reproduced here, although some of the limiting cases are simpler. For example, the author conjectures that $F(u, -\infty)=e^{-u}$, and $F(u, +\infty)=e^{-u^2/4\pi}$ $(u\geqq 0)$, for any $I(t)$ having a spectral density and such that $F(u, I)$ exists.

The paper contains a large number of graphs, and also some discussion of experimental work.

E. Reich (Minneapolis, Minn.)

789:

Czyan, Cze-Pei [Chiang, Tse-pei]. A remark on defining the quantity of information. Teor. Veroyatnost. i Primenen. **3**(1958), 99–103. (Russian. English summary)

Let ξ, η be real-valued random variables. (The author remarks that the following results are easily extended to abstract-valued random variables.) Let P_ξ, P_η, $P_{\xi\eta}$ be the distributions of ξ, η, (ξ, η), respectively. Then P_ξ and P_η are probability measures defined on a certain σ-algebra B of linear sets and $P_{\xi\eta}$ is defined on $B\times B$. The quantity of information J_k $(k=1, 2, 3)$ of ξ contained in η is defined as

$$\sup\sum_1^n P_{\xi\eta}(S_i)\log\frac{P_{\xi\eta}(S_i)}{P_\xi P_\eta(S_i)},$$

where $P_\xi P_\eta$ is the product measure of P_ξ, P_η, $S_1, \cdots, S_n$ are disjunct sets with union the plane, and the supremum is taken over all such decompositions of the plane in which: for J_1, $S_i\in B\times B$; for J_2, S_i is a generalized rectangle, that is, the direct product of two sets in B; for

J_3, S_4 is a rectangle. The following theorems are proved. (i) If $P_{\xi\eta}$ is absolutely continuous with respect to $P_\xi P_\eta$, with density α,

$$J_1 = J_2 = \iint \alpha(x, y) \log \alpha(x, y) P_\xi(dx) P_\eta(dy).$$

(ii) If, in addition, B is the class of linear Borel sets, $J_1 = J_2 = J_3$. These results complete and correct some given by Gel'fand and Yaglom [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 3–52; MR **18**, 980].

J. L. Doob (Urbana, Ill.)

790:

Béthoux, Paul. Discrimination entre plusieurs signaux en télécommunication. C. R. Acad. Sci. Paris **247** (1958), 412–415.

791:

Erohin, V. ε-entropy of a discrete random variable. Teor. Veroyatnost. i Primenen. **3** (1958), 103–107. (Russian. English summary)

Let ξ and ξ' be two discrete chance variables taking the same values x_k, $k=1, 2, \cdots$, and denote their joint distribution by $P(\xi, \xi')$. If $P(\xi, \xi')$ satisfies some condition W of "closeness", the W-entropy is defined as $H_W(\xi) = \inf_{P(\xi,\xi')} J(\xi, \xi')$, where $J(\xi, \xi')$ is the amount of information contained in ξ' about ξ. The paper considers the two conditions W_ξ^1 and W_ξ^2 which specify that $\Pr(\xi \neq \xi') \leq \varepsilon$ and $\Pr(\xi \neq \xi' | \xi') \leq \varepsilon$, where $0 < \varepsilon < 1$. It is shown that the corresponding two W-entropies are equal, and a formula for the common value, the so-called ε-entropy, is given.

G. E. Noether (Boston, Mass.)

792:

Gilbert, E. N. Gray codes and paths on the n-cube. Bell System Tech. J. **37** (1958), 815–826.

Closed paths on an n-dimensional unit cube, useful in the construction of code wheels for encoding angles, and in switching theory, are fully described for $n \leq 4$; a process is given which generates large numbers of paths for $n > 4$.

V. E. Beneš (Murray Hill, N.J.)

CONTROL SYSTEMS

See also 359.

793:

Sawaragi, Yoshikazu; and Sunahara, Yoshifumi. The statistical studies on the response of automatic control systems with a non-linear element of zero-memory type. I. Equivalent gains of non-linear element to an amplitude modulated signal in the presence of Gaussian noise. Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **8** (1958), 95–126.

The authors consider a nonlinear element such that the output signal y is a given function, $f(x)$, of the input signal x. It is assumed that $x(t)$ is a random function of the form $A(t) \cos \omega_0 t + N(t)$, where $A(t)$ is a slowly varying function, ω_0 is a constant, and $N(t)$ is a Gaussian noise. The auto-correlation function of $y(t)$ is calculated by means of standard techniques, depending on the use of the Fourier or Laplace transform of $f(x)$. The results of these calculations are given in detail in the cases of various functions $f(x)$ which are of interest in control system theory. However, the results, in the form of formulae and numerical tables, are too complicated to be exhibited here. The power spectra of $x(t)$ and $y(t)$ are obtained in the usual way, by taking Fourier transforms of the auto-correlation functions; and a comparison of the power spectra enables the authors to define a set of equivalent gains of the non-linear element. In some simple cases the results concerning these gains agree with, but go beyond, the results obtained by other methods of equivalent linearization, such as the describing function method. It appears that this paper is to be followed by two others, in which the theory will be applied to problems concerning nonlinear control systems.

L. A. MacColl (New York, N.Y.)

794:

Pugačev, V. S. Integral canonical representation of random functions and their application to determination of optimal linear systems. Avtomat. i Telemeh. **18** (1957), 971–984. (Russian)

A development is given of the theory of integral canonical representations of random functions and their application to the solution of the problem of determination of an optimal linear operator. As a consequence of the application of this method one obtains a formula for the weight function of an optimal homogeneous linear system in the case of an infinite interval of observation (infinite memory of the system), when the observed random function appears as a result of the passage of white noise through some linear system. In special cases earlier known results of Wiener [Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; MR **11**, 118] and Booton [Proc. I.R.E., **40** (1952), 977–981]. (Author's Summary.)

J. Wolfowitz (Ithaca, N.Y.)

795:

Chakraborti, N. B. On linear delayed control systems. I. Considerations of stability. Indian J. Phys. **32** (1958), 109–123.

Among the simpler linear delayed control systems are those having characteristic equations of the forms $P(p) + Q(p) \exp(-p) = 0$, $P(p) + Q(p) \exp(-p^2) = 0$, where P and Q are polynomials. The author discusses such equations, particularly with respect to the existence or non-existence of roots with positive real parts. Little in the way of general theory is given, the contents of the paper consisting chiefly of a collection of elementary observations concerning the roots, and a detailed discussion of some cases in which P and Q are of low degrees. The author seems to be unacquainted with the extensive work which was done on such equations twenty-five or more years ago. [For references, see R. E. Langer, Bull. Amer. Math. Soc. **37** (1931), 213–239.]

L. A. MacColl (New York, N.Y.)

796:

Davies, D. W. Switching functions of three variables. I. R. E. Trans. **EC-6** (1957), 265–275.

The author deals with the question of minimal representations of switching (two-valued) functions of three variables in terms of switching functions of two variables. The discussion here differs from an earlier one (by the Harvard Computation Laboratory Staff, see; Synthesis of electronic computing and control circuits, Harvard Univ. Press, Cambridge, 1951 [MR **13**, 497]) as well as the comprehensive paper by Slepian [Canad. J. Math. **5** (1953), 185–193; MR **15**, 93]. (See also Pólya [J. Symbolic Logic **5** (1940), 98–103; MR **2**, 65].) The difference lies in permitting complementation of functional values (in the present treatment).

R. M. Baer (Berkeley, Calif.)

797:

Popov, Vasile-Mihai. On relaxation of sufficient conditions of absolute stability. Avtomat. i Telemeh. **19** (1958), 3–9. (Russian. English summary)

The author considers a control system described by the canonical equations

(I) $$\dot{z}_\rho=\lambda_\rho z_\rho+f(\sigma),\quad \rho=1,2,$$
$$\dot{\sigma}=\beta_1 z_1+\beta_2 z_2-rf(\sigma),$$

where β_1, β_2 and r are constant parameters, λ_1 and λ_2 have negative real parts, and $f(\sigma)$, the characteristic of the servomotor, is a function of class A (i.e., (i) f is a continuous function such that $\dot{f}(\sigma)$ has at most a finite number of jumps, (ii) $f(0)=0$, and (iii) $\sigma f(\sigma)>0$). For this system, the author solves the Aizerman problem [Uspehi Mat. Nauk (N.S.) **4(32)** (1949), 187–188; MR **11**, 177]. Let $f(\sigma)=h\sigma$ in (I) and write the characteristic equation of the resulting linear system in the form $\lambda^3+a_1\lambda^2+a_2\lambda+h(r\lambda^2+b_1\lambda+b)=0$. The main result of the paper is that, excluding the case $b_1=0$ and assuming $a_1>0$, $a_2>0$, a set of necessary and sufficient conditions for the asymptotic absolute stability of (I) is: $r\geqq 0$, $b_1>0$, $b_2>0$, $\sqrt{b_2}<\sqrt{(ra_2)}+\sqrt{(a_1b_1)}$. This weakens Lur'e's conditions [Lur'e, A.I., Some nonlinear problems of the theory of automatic regulation, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; MR **15**, 707]:

$$r\geqq 0,\ b_1>0,\ b_2>0,$$
$$\sqrt{(ra_2)}-\sqrt{(a_1b_1)}<\sqrt{b_2}<\sqrt{(ra_2)}+\sqrt{(a_1b_1)}.$$

L. A. Zadeh (New York, N.Y.)

798:

Rozenvasser, E. N. The stability of nonlinear control systems. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 582–585. (Russian)

A sufficient condition, which is extremely complicated and takes several pages to state, is given for stability "in the large" of a control system whose differential equation is

$$\dot{\eta}_k=\sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha+h_k f(\sigma),\ k=1,\cdots,n,$$
$$\sigma=\sum_{s=1}^{n} j_s\eta_s\ (n=5,6).$$

The result follows from a theorem due to Lur'e.

J. P. LaSalle (Baltimore, Md.)

799:

Starikova, M. V. On the investigation of self-oscillations and stability of automatic systems with non-symmetric non-linearity with external exciting force. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 9, 27–32. (Russian)

A nonlinear automatic control system is investigated by the retardation method proposed by E. P. Popov and based on the principle of harmonic balance [N. Kryloff and N. Bogoliuboff, Introduction to non-linear mechanics, Princeton Univ. Press, 1943; MR **4**, 142]. The entire paper is concerned with the determination of the influence of parameters on the system, the effects of a slowly varying perturbation on the amplitude and frequency of the self-excited oscillations, and the choice of parameters which assure that there is no self-excited oscillation and the stability of the equilibrium state. The work was carried out under the direction of B. N. Petov. (Author's summary.) *J. P. LaSalle* (Baltimore, Md.)

800:

***Bellman, R. E.; Glicksberg, I.; and Gross, O. A. Some aspects of the mathematical theory of control processes.** Rand Corporation, Santa Monica, Calif., Rep. No. R-313, xix+244 pp. (1958).

The stated purpose of this book is to provide a taste of the mathematical theory of control processes, both in formulation and in solution. On the whole, this purpose is well achieved. It is to be noted that the authors' concept of a control process is abstract and general, and includes many processes which the typical reader is probably unaccustomed to regard as control processes. On the other hand, many considerations, such as that of "feedback", which are prominent in most discussions of control systems are here disregarded, or are treated only in an involved implicit fashion. It is certain that the authors have formulated a large number of interesting and novel problems, and have solved many of these more or less completely. However, it is difficult to judge the extent to which the theory actually gives useful solutions of realistic practical problems.

The first part of the book is devoted to a class of variational problems, of which the simplest is the following. Two scalar or vector functions, $x(t)$ and $f(t)$, are related by the system of equations $dx/dt=Ax+f$, $x(0)=c$; and it is required to determine f so as to minimize a given functional $F[x, f]$. The intended interpretation of this is as follows. The system $dx/dt=Ax$, $x(0)=c$ is supposed to describe the behavior of some uncontrolled system; in the uncontrolled condition, x fails to conform to some desirable norm; f is a controlling force, intended to improve the performance of the system; and $F[x, f]$ is a conventional measure of the cost of the failure to conform to the norm, and of the cost of applying the controlling force. The various possible forms of $F[x, f]$ lead to many different particular problems. Moreover, the prototype problem can be modified in many ways, by requiring the functions x and f to satisfy subsidiary constraints of various forms. In their discussion of these problems the authors prove various existence, uniqueness, and continuity theorems, and in some instances they give explicit solutions.

The second and third parts of the book are devoted to applications of the theory of dynamic programming and of the theory of games to control processes. The discussion here is extremely concise and rapid, and not at all easy to read.

In general, the make-up of the book is very good. However, in some places reading is made difficult by what appear to be typographical errors or careless and unannounced variations of the formulations of the problems under consideration. In the opinion of the reviewer, the greatest defect lies in the comparative dearth of concrete illustrative examples to aid the reader in clarifying and fixing his knowledge. *L. A. MacColl* (New York, N.Y.)

801:

Loeb, Julien M. Survey of mathematical methods for nonlinear control systems. Trans. A.S.M.E. **80** (1958), 1439–1450.

802:

Löfgren, Lars. Automata of high complexity and methods of increasing their reliability by redundancy. Information and Control **1** (1958), 127–147.

A surface summary of some aspects of Turing machine theory and von Neumann's self-reproducing automata. In addition, there are several calculations of the reduction of error in the transmission of coded information for certain combinations of redundant operations.

R. M. Baer (Berkeley, Calif.)

803:

Krasovskiĭ, N. N. Concerning the theory of optimal control. Avtomat. i Telemeh. **18** (1957), 960–970. (Russian)

The control system is described by the differential equation

$$\frac{dx}{dt} = X(x, t, u), \tag{1}$$

where $x=(x_1, \cdots, x_n)$, $u=(u_1, \cdots, u_r)$, and X is a function on E_{n+1+r} to E_n. Let $x(t_0, x_0, u, t)$ be the solution of (1) satisfying $x(t_0, x_0, u, t_0)=x_0$. The function $u=u(t)$ is called the "steering" function and there is a class of admissable steering functions defined by the restriction

$$g(u, t) \leqq N, \tag{2}$$

where g is a given functional. A moving point (particle) $x=\xi(t)$ in the phase space E_n is given, and the control problem is to hit this moving point in the shortest time; i.e., to select the steering function $u(t)$ so that at some time $t=t_0+T$, $x(t_0, x_0, u, t)=\xi(t)$. The optimal steering function minimizes the transient time T.

In the case where the restriction on the steering function is of the form

$$\int_{t_0}^{t_0+T} G(u(\tau), \tau)d\tau \leqq N,$$

necessary conditions on the optimal control function are obtained by classical calculus of variations techniques. Sufficient conditions are given when the system (1) is linear and $G(u(\tau), \tau)$ is a positive definite quadratic form. The coefficients of the linear system are not assumed to be constant.

Again assuming the system (1) to be linear and with the steering functions u restricted by

$$|u_k(t)| \leqq N, \quad k=1, \cdots, r,$$

the problem of optimal control is reduced to the "L-problem in abstract spaces" and solved by the results of Krein on the L-problem [N. Ahiezer and M. Krein, On certain problems in the theory of moments, Gosudarstv.-Tehn. Izdat. Ukrain., Kharkov, 1938; pp. 173–178]. The optimal system is a bang-bang control system, and this generalizes the results of R. Bellman, I. Glicksberg and O. Gross [Quart. Appl. Math. **14** (1956), 11–18; MR **17**, 1206. See also V. Boltyanskii, R. Gamkrelidze, and L. Pontryagin [Dokl. Akad. Nauk SSSR (N.S) **110** (1956), 7–10; MR **18**, 859]. Examples are given, and the synthesis problem of computing the optimal steering function as a function of the state of the system rather than as a function of time is discussed. *J. P. LaSalle* (Baltimore, Md.)

HISTORY AND BIOGRAPHY

804:

***Menninger, Karl. Zahlwort und Ziffer. Eine Kulturgeschichte der Zahl. Bd. 2: Zahlschrift und Rechnen. 2. neubearbeitete und erweiterte Aufl.** Vandenhoeck und Ruprecht, Göttingen, 1958. xi+314 pp.

The first volume of this book [1957; MR **19**, 517] gave a historical account of the concept of number and of the language in which numbers were, and are, expressed. This second volume deals with the numerical symbols and the elementary computations. It opens with the expression of numbers by fingers, and finger computation; then proceeds to the different methods of indicating numbers on tally sticks, by knots and by written symbols, especially in the Greek, Roman and Arabic way. Then follows the abacus from antiquity to the present sčëty, suan-pan and soroban, with an excursion on counters (jetons, Rechenpfennige). In the last section of the book we find the history of the Hindu-Arabic system; this section ends with a description of Chinese, Japanese and Korean systems of number writing. The wealth of details and the many and excellent illustrations make this book, together with the first volume, the most complete treatise on the subject in existence. It is not only of interest to mathematicians, but to all students of language and of cultural development in general. Here teachers of elementary mathematics should find a generous supply of attractive information, and translators a useful field of endeavor. *D. J. Struik* (Cambridge, Mass.)

805:

Viola, Tullio. Il problema della formazione dei concetti fondamentali della geometria. Sci. Tec. (N.S.) **1** (1957), 1–11.

The article deals with the problem, whether the concepts of mathematics as a science developed historically in the same order as in a child. Specifically, Poincaré maintained correctly, contrary to previous tenets on pure perception of extension and form, that the mathematical concepts of a child develop from experiences relating to motion and deformation. While this was confirmed by psychologists, in particular by observations of Placet extending over 30 years, the order turned out to be the reverse: topological concepts (deformation) precede metric ones (rigid motion) .Without pretending to arrive at definite conclusions, the author is principally concerned with the possibility of obtaining evidence, whether this order also prevailed in the development of mathematical concepts in primitive humanity. *H. Busemann* (Cambridge, Mass.)

806:

Terracini, Alessandro. Cauchy a Torino. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 159–203. (11 plates).

In a lecture given at a centenary commemoration of Cauchy's death (May 22nd 1857) the author recalls the time of his stay at Turin (1832–'33). He came there for political reasons and during a year held the chair of mathematical physics at the University. It was a fertile period, one of the fruits of which was the theory of the convergence of the Taylor series. The author treats Cauchy's relations with the Academy of Sciences at Turin and with the astronomer Plana and the close contact in which he stood in later years to Paolo Ruffini, Felice Chiò and Faà di Bruno. The short article is corroborated by a great number of notes. *E. J. Dijksterhuis* (Bilthoven)

807:

Carruccio, Ettore. I fondamenti dell'analisi matematica nel pensiero di Agostino Cauchy. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 205–216.

On the occasion of the centenary of Cauchy's death the author deals with the rigorous foundations he gave to the calculus by defining exactly the concepts of limit, infinity, continuity etc. It is shown how he convinced the mathematicians of the necessity of always verifying the convergence of infinite series and how he created the theory of the functions of complex variables. Finally the reader is given an impression of Cauchy's ideas on the limits of the domain where mathematical reasoning is applicable. *E. J. Dijksterhuis* (Bilthoven)

808:

Truesdell, C. Neuere Anschauungen über die Geschichte der allgemeinen Mechanik. Z. Angew. Math. Mech. **38** (1958), 148–157.

Under the influence of E. Mach the value of 18th-century mechanics has been grossly underrated. Especially the work of Euler is still not sufficiently appreciated. He laid the foundations of a really general mechanics, in which not only material points, but rigid, elastic and fluid systems as well were considered. The author deals with hydrodynamical researches of Johann and Daniel Bernoulli, d'Alembert, Clairaut and Euler, the foundations of the theory of elasticity, and Cauchy's field equations for continuous media. He finally points out that rational mechanics, though based on empirical facts, is not an experimental science, but a product of the mathematical way of thinking.

E. J. Dijksterhuis (Bilthoven)

809:

Rychlik, Karel. Un manuscrit de Cauchy aux archives de l'académie tchécoslovaque des Sciences. Czechoslovak Math. J. **7(82)** (1957), 479–481. (Russian summary)

A shortened version of the Czech paper [Časopis Pest. Mat. **82** (1957), 227–228] listed in MR **19**, 826. The manuscript in question is holograph, contains 72 pages and was dated by Cauchy himself "Prague 1835". Its title is "Memoire sur l'intégration des équations differentielles". Because of the departure of Cauchy from Prague in 1836, this manuscript was not printed, as had been intended, in the Proceedings of the Société Royale des Sciences de Bohême. The article later formed part of Cauchy's "Exercises d'Analyse et de Physique Mathématique," Tome 1, Paris, 1840.

810:

Oettel, H. Zum 250. Todestag von Tschirnhaus. Ein Mathematiker des Barock. Math. Naturwiss. Unterricht **11** (1958), 194–198.

A general and scientific biography with one photograph.

811:

Anonymous. Obituary: Salomon Lubelski. Acta Arith. **4** (1958), 1–2.

A sixteen-line biography with a bibliography of 19 entries.

GENERAL

See also 136.

812:

***Archbold, J. W. Algebra.** Sir Isaac Pitman and Sons, Ltd., London, 1958. xix+440 pp. 45s.

This lively text is prepared for the specific purpose of including within the covers of a single volume all the material on Algebra mentioned in the syllabuses for certain mathematical Degrees of the University of London. The first half of the book (Chapters 1–15) deals with traditional topics, starting with Natural Numbers, more than usual space being given to Geometry of Complex Numbers (the traditional subjects of Continued Fractions and of Indeterminate Analysis are omitted), while the latter half covers the elements of Linear Algebra, confined to cases with ground field commutative and "without characteristic." Save in connection with such topics as Horner's method, and Rolle's Theorem, all questions of convergence are left to other courses. The text is particularly rich in interesting and challenging exercises here assembled for the first time. The work is accurate, adequate, and interesting. One notes, however, the absence of any discussion of such basic notions as "set", "Cartesian product", "function", etc. The computation of π as a decimal is brought to date (purportedly) by "Ferguson [Math. Gazette, **30** (1946) 89], however, disputes Shank's figures after the 520th place." Similarly e is stated as having been calculated "to 346 decimal places" (no date mentioned). *A. A. Bennett* (Carbondale, Ill.)

813:

Stone, Marshall H. The future of mathematics. J. Math. Soc. Japan **9** (1957), 493–507.

An address delivered before the Mathematical Society of Japan, May 21, 1956.

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